SOLAR SYSTEM MAGNETOHYDRODYNAMICS

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I. THE MACROSCOPIC EQUATIONS OF A PLASMA

I.1. From Particles to Fluids

Continuum mechanics is that branch of physics that treats the motions of infinitely deformable matter. It embraces hydrodynamics, aerodynamics, magnetohydrodynamics (MHD), and magnetogasdynamics. The first two differ in that the former is incompressible and the latter compressible fluid dynamics. The prefix, magneto, signifies the addition of the ponderomotive force(colloquially called the J-cross-B force) to the usual pressure gradient, gravitational and viscous forces of fluid dynamics. Magnetofluid mechanics applies to fluids that can carry electrical currents, such as liquid metals and plasmas. Our interest in Solar System MHD is confined to the latter.

In contrast to particle mechanics and rigid body mechanics, the mass on which the forces act in continuum mechanics is distributed throughout some volume of space, and any portion of the distributed mass can in principle move in an arbitrary direction or manner relative to any other portion. This has as one of its consequences the necessity of including the coordinates of space among the independent variables; and thus the equations of continuum mechanics are partial differential equations in space and time rather than ordinary differential equations in time.

The dependent variables of hydro- and aerodynamics are velocity (a vector), mass density (a scalar) and pressure (usually a scalar but sometimes a tensor). In magnetohydro- and magnetogasdynamics, one adds the magnetic field, and in magnetogasdynamics the occasions when the pressure must be treated as a tensor are more common.

For those who are concerned with applying the equations of continuum mechanics to describe the behavior of the oceans or the atmosphere, there are instruments that measure directly the required dependent variables, such as wind and pressure. The winds of space generally greatly exceed those which occur in the troposphere. The solar wind traverses in one second a three hour track of the hurricane wind. Yet the plasmas of space are so tenuous that the pressures they exert even with the full force of their winds are immeasurable by direct pressure sensing devices. The instruments for measuring the properties of plasmas in space take advantage of the fact that the individual particles that comprise the "continuum" there are electrically charged. These instruments either detect the charges of the particles individually or collectively as an electrical current. There is then a gap between the dependent variables that enter into the equations of continuum mechanics and the measured variables. The gap must be filled by mathematical processing of the data, in order to go from information about the individual particles to information about the fluid that they constitute in their bulk. There is a formalism that treats this problem specifically. This formalism begins with the phase space density.

I.2 Phase Space Density

It is necessary to specify the physical state of our collection of individual particles precisely. The physical state of a particle is given by its mass and charge, which are known constants (we treat only the non-relativistic case), its position $\vec{x}$ and its velocity $\vec{v}$. Therefore six independent numbers are needed to fix the physical state of a particle, $(x_1, x_2, x_3, v_1, v_2, v_3)$, where the subscripts denote the three independent orthogonal axes of configuration and velocity space. We may think of the six numbers as being the coordinates of the particle in a six-dimensional space, called <u>phase space</u>.

If we were so to locate each particle of our plasma, we would build up a non-homogeneous distribution of points occupying some portion of phase space. In virtually all problems that are concerned with the behavior of space plasmas on a macroscopic scale, the number of particles is truly enormous. Thus, it is meaningful to speak of a density of points in phase space, $f(x,v)$, which is a function of the six independent variables, and when multiplied by the six-dimensional volume element d^3xd^3v gives the number of particles with coordinates that lies in the range $(x_1, x_2, x_3, v_1, v_2, v_3)$ to $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3, v_1 + dv_1, v_2 + dv_1, v_3 + dv_3)$. Since we must allow for the change of the distribution with time, the <u>phase space density</u> is actually a function of seven independent variables, $f(\vec{x}, \vec{v}, t)$. The quantity f as we have defined it is also referred to as the <u>single particle distribution function</u>.

I.3 The Continuity Equation in Phase Space

The number density in six-dimensional phase space is $f(\vec{x}, \vec{v})$, to which we can formally assign a current, viz. $\vec{x}\, f + \vec{v}\, f$, analogous to the particle current or flux given by $n\vec{v}$ in usual language. Applying <u>the generalized conservation principal</u>

$$- \frac{\partial Q}{\partial t} = \nabla_n \cdot (Q\vec{v}_n) \qquad \text{(I.1)}$$

where Q is any density, ∇_n is the divergence in n-space and $\vec{V}_n$ is n-space velocity, to f we find

$$-\frac{\partial f}{\partial t} = \frac{\partial}{\partial x_i}(\dot{x}_i f) + \frac{\partial}{\partial v_i}(\dot{v}_i f) \tag{I.2}$$

in which the summation convention is used (i.e. $a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$). Since $\dot{x}_i = v_i$ and $\dot{v}_i = F_i/m$, (I.2) can be written

$$-\frac{\partial f}{\partial t} = \nabla \cdot (\vec{v} f) + \nabla_v \cdot (\frac{\vec{F}}{m} f) \tag{I.3}$$

where $\nabla_v = \frac{\partial}{\partial v1} \hat{e}_1 + \frac{\partial}{\partial v2} \hat{e}_2 + \frac{\partial}{\partial v3} \hat{e}_3$, and $\hat{e}_i$ is the unit vector corresponding to the ith coordinate.

The only forces to which our individual particles are subject are the gravitational and Lorentz forces

$$\vec{F} = m\vec{g} + q\vec{E} + q\vec{v} \times \vec{B} \tag{I.4}$$

where q is the electrical charge on the particle. (The only other forces at this fundamental, single particle level are the weak and strong forces of nuclear interaction, with which we are not concerned.)

We now proceed to reduce (I.3) to a more specific form. Since x and v are independent variables, we may set $\nabla \cdot (\vec{v} f) = \vec{v} \cdot \nabla f$. Also it is evident that $\nabla_v \cdot (m\vec{g}) = \nabla_v \cdot (q\vec{E}) = 0$ since none of m, g, q, or E depends on v. Finally

$$\nabla_v \cdot q(\vec{v} \times \vec{B}) = q\vec{B} \cdot (\nabla_v \times \vec{v}) - q\vec{v} \cdot (\nabla_v \times \vec{B})$$

But both curls vanish, therefore

$$\nabla_v \cdot (\frac{\vec{F}}{m} f) = \frac{\vec{F}}{m} \cdot \nabla_v f + f \nabla_v \cdot \frac{\vec{F}}{m}$$

and

$$-\frac{\partial f}{\partial t} = \vec{v} \cdot \nabla f + \frac{\vec{F}}{m} \cdot \nabla_v f \tag{I.5}$$

I.4 The Boltzmann Equation

We have until now ignored the effects of collisions. We can include them by noting that the effect of a collision is to relocate a particle along the velocity axes of phase space "instantaneously". Therefore we may represent the effect of collision by adding a separate collision term to (I.5) as follows:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \frac{\vec{F}}{m} \cdot \nabla_v f = (\frac{\partial f}{\partial t})_{coll} \tag{I.6}$$

This form of the conservation equation is referred to as the Boltzmann equation.

I.5 Liouville's theorem:

In the absence of collisions, (I.6) becomes

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f + \frac{\vec{F}}{m}\cdot\nabla_V f \equiv \frac{df}{dt} = 0 \qquad (I.7)$$

with the meaning that f is constant as it is convected with the particles.

I.6 Forming the Macroscopic Variables

We consider a two component gas composed of positive and negative ions, which we take to protons and electrons in practice. The charge, q, is then either + e or −e, where e is the value of the electronic charge. The macroscopic or bulk quantities are then formed from the phase space density as follows:

mass density $\rho = \sum_a m_a \int f_a \, d^3 v$ (I.8)

charge density $\rho_c = \sum_a e_a \int f_a \, d^3 v$ (I.9)

bulk velocity $\vec{V} = \frac{1}{\rho} \sum_a m_a \int \vec{v} f_a \, d^3 v$ (I.10)

current density $\vec{J} = \sum_a e_a \int \vec{v} f_a \, d^3 v$ (I.11)

pressure tensor $P_{ij} = \sum_a m_a \int (v_i - V_i)(v_j - V_j) f_a \, d^3 v$ (I.12)

internal energy $u = \frac{1}{2}\,\mathrm{Trace}\,(P) = \frac{1}{2}\sum_a m_a \int (\vec{v} - \vec{V})^2 f_a \, d^3 v$ (I.13)

heat flux vector $\vec{q} = \frac{1}{2}\sum_a m_a \int (\vec{v} - \vec{V})^2 (\vec{v} - \vec{V}) f_a \, d^3 v$ (I.14)

where a is an index with values 1 and 2 for the two components.

One can see that the macroscopic variables are constructed out of appropriate velocity moments of f. Mass and charge density result from the zeroth velocity moment, bulk velocity and electrical current density from the first moment, pressure and internal energy are related to the second moment, and the heat flux vector to the third moment.

I.7 Derivation of the Macroscopic Equations

If we were given continuous measurements of f for all regions of space, or if f were computed from the Boltzmann equation and a comprehensive set of initial measurements, the preceding relations could be used to determine any of the macroscopic quantities of interest at any time and place of interest. Of course, neither procedure is practical, and even if it were, it would be wasteful, since f contains information on the distribution of the particles in velocity space, which is completely ignored in the macroscopic variables. Phase space density is a function of seven variables, whereas the macroscopic quantities depend on only four. There is a great advantage then, if one is interested only in the macroscopic quantities, to find equations to predict subsequent values of the macroscopic variables from some initial set which is specified by a model or obtained from a definite set of measurements of f. What we require is the macroscopic analog of the Boltzmann equation for f. Since in the macroscopic description there are a number of dependent variables, it is evident that more than one such analog is needed. We have noted that the macroscopic variables are obtained from f through the operation of taking velocity moments. It is natural then to seek the desired macroscopic equations by taking velocity moments of the Boltzmann equation.

With the foregoing motivation, we may proceed in a strictly formal manner, and define the n-th moment operator M_n operating on any quantity or expression designed by the symbol $[\cdots]$ by

$$M_n[\cdots] = \sum_a \frac{m_a}{n!} \int v_i^n [\cdots]_a d^3v, \quad n = 0,1,2 \cdots \qquad \text{(I.15)}$$

where the summation convention is implied, i.e. $(v_i)^2 = v^2$, $(v_i)^3 = v^2 v_i$, etc. We apply this operator to the Boltzmann equation in the form which is most convenient for our purpose

$$\frac{\partial f}{\partial t} + \nabla \cdot (f \vec{v}) + \nabla_v \cdot \left(\frac{\vec{F}}{m} f\right) = \left(\frac{\partial f}{\partial t}\right)_{coll} \qquad \text{(I.16)}$$

Since the number of particles, the momentum, and the kinetic energy are conserved in collisions, we have as a direct consequence

$$M_n \left[\left(\frac{\partial f_a}{\partial t}\right)_{coll}\right] = 0 \qquad n = 0,1,2 \text{ only} \qquad \text{(I.17)}$$

Mass $n = 0$: Applying (I.15) to (I.16) with $n = 0$ and with definitions (I.8) and (I.10) there results immediately the continuity equation.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \qquad \text{(I.18)}$$

where it has been assumed that $\frac{\vec{F}}{m} f$ vanishes sufficiently rapidly at infinity in velocity space. Equation (I.18) is the expression for the conservation of mass as can be seen by integrating over an arbitrary volume element

$$\int \frac{\partial \rho}{\partial t} d^3x = \frac{d}{dt} \int \rho \, d^3x = \frac{dM}{dt} = - \oint_s \rho \vec{V} \cdot \hat{n} \, d^2x \qquad \text{(I.19)}$$

where s is the surface of the volume. The integrand is the mass flux per unit area. Thus, the rate of change of mass in the volume dM/dt is the net rate at which it is entering or leaving the volume.

The continuity equation is often written in terms of the convective derivative (also known as the total derivative and the substantial derivative) defined by

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{V}\cdot\nabla \tag{I.20}$$

It represents the time rate of change in a frame of reference moving with the fluid. Then (I.19) becomes

$$\frac{d\rho}{dt} + \rho \; \nabla\cdot\vec{V} = 0 \tag{I.21}$$

For future reference we note that (I.21) is sometimes used as an equation for replacing $\nabla\cdot\vec{V}$ by an expression involving only ρ, namely

$$\nabla\cdot\vec{V} = -\frac{1}{\rho}\,\frac{d\rho}{dt} = -\frac{d\ell n\,\rho}{dt} = \frac{d}{dt}\,\ell n(\frac{1}{\rho}) = \rho\,\frac{d}{dt}\,(\frac{1}{\rho}) \tag{I.22}$$

Momentum $n = 1$: After several steps of fairly straightforward algebra, the first moment of the Boltzmann equation can be cast in the form of a momentum equation also known as an Euler Equation which written in component notation becomes

$$\rho \;\frac{dV_i}{dt} = -\frac{\partial P_{ij}}{\partial\, x_j} + \rho_c\, E_i + (\vec{J} \times \vec{B})_i + \rho g_i \tag{I.23}$$

in which the second term is the divergence of the pressure tensor defined by eq. (I.12).

The right hand side of the Euler equation enumerates the various forces that can accelerate an element of plasma, namely imbalances in the pressure forces acting on its surface, the electrostatic force, the ponderomotive force and the gravitational force. In the absence of interactions with neutral particles, there are no other forces. The viscous force is contained in the divergence of the pressure tensor.

Viscosity arises when a fluid is in non-uniform motion. Momentum of relative motion is exchanged between adjacent fluid elements through the cross migration of the individual constituent particles in the course of their thermal wandering. The viscous force therefore must depend explicitly on velocity gradients and on parameters that characterize the degree of cross migration. The part of the pressure tensor that contains the effect of viscosity is called the viscous stress tensor. In the highly collisional domain the viscous stress tensor has the form

$$S_{ij} = -\eta \;(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} - \frac{2}{3}\;\nabla\cdot\vec{V}\,\delta_{ij}) - \zeta\;\nabla\cdot\vec{V}\,\delta_{ij} \tag{I.24}$$

where η and ζ are the coefficients of viscosity, both of which are positive numbers.

In the collisional domain for which (I.24) is an appropriate representation of S_{ij}, the thermal motions of the particles are isotropic and the non-viscosity, purely pressure part of the pressure tensor is also isotropic. The entire pressure then has the form

$$P_{ij} = p\,\delta_{ij} + S_{ij} \qquad \text{(I.25)}$$

where p is the scalar pressure. The force that results from taking the divergence of (I.25) can be written as explicit pressure-force and viscous-force terms to the right hand side of the Euler equation. These are

$$\nabla\cdot\overleftrightarrow{P} = \vec{\nabla}p + \eta\nabla^2\vec{V} + (\zeta + \frac{1}{3}\eta)\,\nabla(\nabla\cdot\vec{V}) \qquad \text{(I.26)}$$

where η and ζ have been treated as constants, as is normally done for mathematical convenience, although they are functions of pressure and temperature. In ordinary fluid dynamics the terms in (I.26) alone constitute the right hand side of the Euler equation,in which form it is known as the Navier-Stokes equation.

In a large number of solar system applications, the plasma must be treated as basically collisionless. Then the magnetic field greatly inhibits cross migration in the direction perpendicular to itself, but not in the parallel direction. The viscous stress tensor then has an influence mainly on velocity gradients parallel to the field. (See Rossi and Olbert, 1970, for a discussion of the general viscous stress tensor in a magnetized plasma). As a general observation, viscosity has not yet played a major role in discussions of solar system plasmas, although there are occasional exceptions to this statement in solar wind theory and in the theory of the solar wind coupling to planetary magnetospheres and ionospheres. The viscous stress tensor and viscous forces therefore will be omitted in the remainder of the chapter.

When collisions are sufficiently rare that charged particles gyrate around the magnetic field many times before colliding, the pressure tensor tends to become anisotropic. With reference to a local cartesian coordinate system which has its z-axis parallel to the field, the gyromotion acts to isotropy the pressure in the xy-plane. The component that relates to the z-axis is decoupled to a degree that depends on the actual extent of collisions. Two scallars are therefore necessary to represent the two partially or completely decoupled motions. The pressure tensor then has the form

$$P_{ij} = (p_{\parallel} - p_{\perp})\, b_i b_j + p_{\perp}\,\delta_{ij} \qquad \text{(I.27)}$$

or in vector notation

$$\overleftrightarrow{P} = (p_{\parallel} - p_{\perp})\,\hat{b}\,\hat{b} + p_{\perp}\,\overleftrightarrow{I} \qquad \text{(I.28)}$$

where $\hat{b}$ is a unit vector parallel to the magnetic field, $\vec{B}$, and $\overleftrightarrow{I}$ is the unit, diagonal tensor. To verify that (I.27) and (I.28) express the properties attributed to the pressure tensor in this case, scalar multiply (I.28) with unit vectors that are perpendicular and parallel to $\vec{B}$. Thus, if we once again let $\hat{b} = \hat{z}$,

$$\hat{x} \cdot \overleftrightarrow{P} \cdot \hat{x} = \hat{y} \cdot \overleftrightarrow{P} \cdot \hat{y} = p_{\perp} \qquad \text{(I.29)}$$

$$\hat{z} \cdot \overleftrightarrow{P} \cdot \hat{z} = p_{\parallel} \qquad \text{(I.30)}$$

Equations (I.29) and (I.30) define the scalars $p_{\perp}$ and $p_{\parallel}$.

<u>Energy n = 2</u>: The second moment of the Boltzmann equation leads after several intermediate but straightforward steps to an expression relating energy densities in the plasma

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho V^2 + u\right) + \frac{\partial}{\partial x_j}\left[\left(\frac{1}{2}\rho V^2 + u\right) V_j + P_{jk} V_k + q_j\right] = J_j E_j + \rho V_j g_j \qquad \text{(I.31)}$$

Inspection of equations (I.18), (I.23), and (I.31) shows that they are prognostic equations for mass density, ρ, material momentum density, $\rho\vec{V}$, and the total kinetic energy density, $\frac{1}{2}\rho V^2 + u$, that is, they are equations that specify the time rates of change of these quantities. Since we know that mass momentum and energy are subject to conservation laws, it should be possible to recast these expressions into the general conservation form given by equation (I.1). To achieve the desired restructuring we need the field equations for the electromagnetic and gravitational fields.

I.8 The Field Equations

The electric, magnetic and gravitational fields are related to the macroscopic quantities defined in I.6 by <u>Maxwell's equations</u> and <u>Newton's equations</u>.

$$\nabla \cdot \vec{E} = \rho_c/\varepsilon_o \qquad \text{(I.32)}$$

$$\nabla \cdot \vec{B} = 0 \qquad \text{(I.33)}$$

$$\nabla \times \vec{E} = -\,\partial\vec{B}/\partial t \qquad \text{(I.34)}$$

$$\nabla \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o\, \partial\vec{E}/\partial t \qquad \text{(I.35)}$$

and

$$\nabla \cdot \vec{g} = -4\pi G \rho \qquad \text{(I.36)}$$

$$\nabla \times \vec{g} = 0 \qquad \text{(I.37)}$$

$\varepsilon_o = 8.854 \times 10^{-12}$ Farad/m, $\mu_o = 4\pi \times 10^{-7}$ Henry/m

$1/\varepsilon_o\mu_o = c^2$, $c = 2.998 \times 10^8$ m/sec.

$G = 6.670 \times 10^{-11}$ m^3/kg-sec.

The fields exert stresses and possess momentum and energy. The stresses are given by the Maxwell stress tensor

$$T_{ij} \equiv \varepsilon_o E_i E_j + \frac{1}{\mu_o} B_i B_j - \left(\frac{\varepsilon_o}{2} E^2 + \frac{1}{2\mu_o} B^2\right) \delta_{ij} \tag{I.38}$$

and the gravitational stress tensor

$$\Gamma_{ij} = -\frac{1}{4\pi G} \left(g_i g_j - \frac{1}{2} g^2 \delta_{ij}\right) \tag{I.39}$$

The divergences of the stress tensors can be seen to correspond to the body forces on the plasma that appear in equation (I.23)

$$\frac{\partial T_{ij}}{\partial x_j} = \varepsilon_o \mu_o \frac{\partial S_i}{\partial t} + \rho_c E_i + (\vec{J} \times \vec{B})_i \tag{I.40}$$

$$\frac{\partial \Gamma_{ij}}{\partial x_j} = \rho g_i \tag{I.41}$$

where

$$\vec{S} \equiv \frac{\vec{E} \times \vec{B}}{\mu_o} \tag{I.42}$$

is the Poynting vector.

The energy densities of the fields are given by $\varepsilon_o E^2/2$, $B^2/2\mu_o$, and $\rho\phi$ where ϕ is the gravitational potential defined by

$$\vec{g} = -\nabla \phi \tag{I.43}$$

The flux of energy in the electromagnetic field is the Poynting vector, as can be identified from Poynting's theorem

$$\nabla \cdot \vec{S} = \nabla \cdot \frac{\vec{E} \times \vec{B}}{\mu_o} = \frac{1}{\mu_o} [\vec{B} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{B})]$$

which becomes upon substitutions from Maxwell's equations

$$\frac{\partial}{\partial t} \left(\frac{\varepsilon_o}{2} E^2 + \frac{1}{2\mu_o} B^2\right) + \nabla \cdot \vec{S} = -\vec{E} \cdot \vec{J} \tag{I.44}$$

The term on the right hand side of this expression, $- \vec{E} \cdot \vec{J}$, is the electromechanical energy conversion term. It represents a source or a sink of electro-magnetic energy, depending on the sign of $\vec{E} \cdot \vec{J}$.

The flux of gravitational energy can be constructed in the usual way, by multiplying the gravitational energy density, $\rho\phi$, by the velocity, $\vec{V}$. The choice is verified by taking the divergence of the trial expression and showing that it satisfies the appropriate conservation equation

$$\nabla \cdot (\rho \vec{V} \phi) = - \phi \frac{\partial \rho}{\partial t} - \rho \vec{V} \cdot \vec{g}$$

or

$$\frac{\partial}{\partial t} (\rho \phi) + \nabla \cdot (\rho\phi\vec{V}) = - \rho\vec{V}\cdot\vec{g} + \rho \frac{\partial \phi}{\partial t} \qquad \text{(I.45)}$$

On the right hand side the combination $- \rho \vec{V}\cdot\vec{g}$ is recognizable as the gravitational-mechanical energy conversion term. The second term, while formally present, in practice is absent since the gravitational fields are assumed to be constant.

I.9 The Conservation Equations

The continuity equation (I.18) is already in the form of a conservation equation for mass. It remains to find similar expressions for momentum and energy. Combination of equations (I.23) (I.40) and (I.41) produces the desired result for momentum

$$\frac{\partial}{\partial t} (\rho V_i + \varepsilon_o \mu_o S_i) + \frac{\partial}{\partial x_j} (A_{ij}) = 0 \qquad \text{(I.46)}$$

where

$$A_{ij} \equiv P_{ij} + \rho V_i V_j - T_{ij} - \Gamma_{ij} \qquad \text{(I.47)}$$

is the grand momentum stress tensor. It contains the pressure, Maxwell and gravitational stress tensor, which have already been introduced, and a new term, $\rho V_i V_j$ which is called the Reynold's stress tensor. It is the macroscopic analog of the viscous stress tensor (which at this level of exposition is still latent in the pressure stress tensor). It represents momentum in the i-direction, ρV_i, that transferred by the velocity component, V_j, across a plane which has its normal in the j-direction. In turbulence theory, the analog between the transport of momentum by random eddys on the macroscale and by the random motions of molecules on the microscale is particularly appropriate, and it is where the Reynolds stress tensor as an explicit concept was developed.

Equation (I.46) shows that the combination $\varepsilon_o \mu_o \vec{S} = \vec{S}/c^2$ is the electromagnetic momentum [redacted] density. In virtually all solar system applications $\vec{S}/c^2$ is completely negligible compared to $\rho\vec{V}$, and it can be ignored in the conservation equation with impunity. The validity of this statement will be demonstrated in Section II.2.

The expression for energy conservation results from combining equations (I.31), (I.44) and (I.45).

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho V^2 + u + \frac{\varepsilon_o}{2}E^2 + \frac{1}{2\mu_o}B^2 + \rho\phi\right) + \nabla \cdot \left[\left(\frac{1}{2}\rho V^2 + u + \overleftrightarrow{P}\right)\cdot \vec{V} + \vec{q} + \vec{S} + \rho\vec{V}\phi\right] = \rho\frac{\partial \phi}{\partial t} \qquad (I.48)$$

We recognize the terms acted on by the time derivative to be the various forms of energy densities. The expression shows our list of energy densities to be complete. The terms acted on by the divergence operator represent processes by which energy can enter and leave a volume through its surface. It is clear that $(\frac{1}{2}\rho V^2 + u + \rho\phi)\vec{V}$ is the flux of the total kinetic energy density $(\frac{1}{2}\rho V^2 + u)$ plus the gravitational energy density through the surface. The combination $\overleftrightarrow{P}\cdot\vec{V}$ is the work done on the volume by the pressure stress as a result of the motion $\vec{V}$ across the surface. In the case of a scalar pressure, the combination $u + p$ which enters the conservation equation as an effective convected energy density is referred to as the <u>enthalpy</u> of the flow.

The heat flux vector, $\vec{q}$, is seen to have the character expected of it, namely the flow of energy across a surface, or more precisely, it is an energy flux density. Note that we may have the transfer of kinetic energy in the absence of convection if $\vec{q} \neq 0$ and $\vec{V} = 0$. The transfer in this case results completely from directional asymmetries in the single particle distribution function, that is, the heat flux represents a microscopic transport process. The macroscopic medium itself is not moving, but energy is flowing in or through it.

The flow of electromagnetic energy is represented solely by the Poynting vector, as claimed earlier. In contrast to the electromagnetic momentum flux density, the electromagnetic energy flux density is often comparable to or greater than the largest of the other terms in the equation. It must be retained in solar systems applications of MHD.

<u>Charge</u> To complete the discussion of conservation equations, we derive the equation for the conservation of charge. For this we define a new operator

$$Q_n[\cdots] \equiv \sum_a \frac{e_a}{n!}\int d^3v\, v_i^n\, [\cdots]_a \qquad (I.49)$$

Comparison with equation (I.15) shows that we can distinguish between the two operators by calling the first the "mass-velocity moment operator" and the second the "charge-velocity moment operator." For the zeroth charge-moment of the Boltzmann equation we find

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot \vec{J} = 0 \qquad (I.50)$$

The collision term has again been set equal to zero since charge is conserved in collisions. This result is consistent with Maxwell's equations from which it can also be derived. The divergence of (I.35) gives

$$0 = \mu_o \nabla \cdot \vec{J} + \mu_o \varepsilon_o \frac{\partial}{\partial t} \nabla \cdot \vec{E}$$

which with (I.32) is seen to be the same as (I.50).

The first charge-moment of the Boltzmann equation will be discussed in Section II.1

I.10 Inclusion of Neutral Particle Interactions

Ionization of neutrals, charge exchange between ions and neutrals and the production of neutrals from ions by recombination are important processes in solar system plasma. Examples of interacting bodies of plasmas and neutrals include the solar wind and the interstellar medium, the solar wind and comets, the solar wind and planetary atmospheres, the ionized and neutral components of satellite torii in the magnetospheres of Jupiter and Saturn, and planetary ionospheres and the neutral atmospheres in which they occur. The effects of neutrals can be included in the MHD equations as source and loss terms on the right hand sides of the conservation equations just derived.

Designate the source and loss terms corresponding to the equations for the conservation of mass momentum and energy by Q_M, Q_P, and Q_E respectively. Symbolically Q_M can be written as

$$Q_M = \sum_k \frac{(m_n)_k n_k}{(\tau_i)_k} - \frac{\rho}{\tau_r} + \sum_{k,\ell} \frac{[(m_n)_k - (m_i)_\ell]\, n_k}{(\tau_{ex})_{k\ell}} \qquad (I.51)$$

in which n_k is the number density of the k-th species of neutral particle which gives rise to an ion of mass $(m_n)_k$ in a characteristic ionization time $(\tau_i)_k$. In general ionization can result from photoionization, with time scale τ_{ph}, and electron impact ionization with time scale $\tau_{e\ell}$. Thus

$$\frac{1}{(\tau_i)_k} = \frac{1}{(\tau_{ph})_k} + \frac{1}{(\tau_{e\ell})_k} \qquad (I.52)$$

The second term in (I.51) represents loss by recombination, with time scale τ_r. The time scale itself depends on the plasma density ρ

since the number of electron encounters per second a given ion experiences increases with the density of the plasma. The third term gives the contribution to the spontaneous increase or decrease in mass density by charge exchange between neutral species k and ionized species ℓ. Since in a charge exchange interaction, the plasma gains the mass of the former neutral and losses the mass of the former ion, the net gain is the difference in the two masses, which can be negative. If an ion species charge exchanges with its own neutral species, there is no net change in mass density. In this expression the time scale $(\tau_{ex})_{k\ell}$ is referenced to the neutral species and therefore implicitly contains the density of the ion species.

Corresponding expressions can now readily be written for Q_P and Q_E. The composite momentum of the newly created ions adds to Q_P and the composite momentum of the newly lost ions subtracts from it. Similarly the energy of the new ions adds to Q_E and the energy of the lost ions subtracts from it. It is important in this case also to include the energies of the electrons that are either gained or lost in the process. In connection with Q_E it should be noted that at least one plasma in the solar system, the Io torus, exhibits a substantial loss of energy by electromagnetic radiation which must be included in Q_E.

The purpose of this section is to indicate the existence of an important class of phenomena in solar system plasmas involving interactions with neutrals. To treat these situations the equations of ideal MHD need to be modified in the way outlined here.

I.11 The Prognostic Equation for Scalar Pressure

The equations for mass, momentum, and energy can be combined to produce a simple prognostic equation for pressure. We will treat the case of a scalar pressure first. Then there is one such equation. Later we retain the tensor character of pressure that is appropriate to a magnetized plasma. Two equations are then required. The pressure equation can replace one of the original three, and it is usual to exchange it for the most complex of these, the energy equation.

In the case of scalar pressure (I.25 without S_{ij}) the internal energy (I.13) is given by

$$u = \frac{3}{2} p \qquad \text{(I.53)}$$

The scalar product of the Euler equation (I.23 with scalar pressure) with $\vec{V}$ can be manipulated into the following form with the use of the continuity equation (I.18)

$$\frac{d}{dt} \left(\frac{1}{2} \rho V^2\right) + \frac{1}{2} \rho V^2 \, \nabla\cdot\vec{V} + \vec{V}\cdot\nabla p = \rho_c \, \vec{V}\cdot\vec{E} + \vec{V}\cdot(\vec{J}\times\vec{B}) + \rho V\cdot g \qquad \text{(I.54)}$$

With the substitution (I.53), the energy equation (I.31) can be manipulated into a similar form

$$\frac{d}{dt}\left(\frac{1}{2}\rho V^2\right) + \frac{1}{2}\rho V^2\,\nabla\cdot\vec{V} + \frac{\partial}{\partial t}\left(\frac{3}{2}p\right) + \vec{V}\cdot\nabla\left(\frac{5}{2}p\right) + \frac{5}{2}p\,\nabla\cdot\vec{V}$$

$$= -\nabla\cdot\vec{q} + \vec{J}\cdot\vec{E} + \rho\vec{V}\cdot\vec{g} \qquad \text{(I.55)}$$

Subtracting (I.54) from (I.55), and eliminating $\nabla\cdot\vec{V}$ by use of (I.22), we find after some rearranging of the right hand side

$$\frac{d}{dt}\left(\frac{3}{2}p\right) - \frac{5}{2}\frac{p}{\rho}\frac{d\rho}{dt} = -\nabla\cdot\vec{q} + (\vec{J}-\rho_c\vec{V})\cdot(\vec{E} + \vec{V}\times\vec{B}) \qquad \text{(I.56)}$$

Equation (I.56) can now be rewritten in a more revealing form

$$\frac{d}{dt}\left(\frac{p}{\rho^{5/3}}\right) = \frac{-\nabla\cdot\vec{q} + \vec{j}\cdot\vec{E}^*}{\frac{3}{2}\rho^{5/3}} \qquad \text{(I.57)}$$

in which the symbols $\vec{j}$ and $\vec{E}^*$ are defined by

$$\vec{j} \equiv \vec{J} - \rho_c\vec{V} \qquad \text{(I.58)}$$

$$\vec{E}^* = \vec{E} + \vec{V}\times\vec{B} \qquad \text{(I.59)}$$

Before describing the interpretation and use of the primary result, equation (I.57), we digress briefly to note the meaning of these newly defined current and electric field variables. The current density $\vec{j}$ is the current that arises from the relative motion of positive and negative charges in equal numbers, and can be referred to as the <u>conduction current density</u>. The conduction current density is to be distinguished from the <u>convection current density</u>, $\rho_c\vec{V}$, which is simply the transport with the fluid of any excess positive or negative charge in the plasma. The total current density, $\vec{J}$, is then the sum of the conduction and convection current densities.

From electromagnetic theory, we know that in the non-relativistic limit and in the presence of a magnetic field, $\vec{B}$, the electric field, $\vec{E}^*$ in a frame of reference moving with velocity $\vec{V}$ relative to a frame in which the electric field is $\vec{E}$ is given precisely by equation (I.59). Thus, <u>$\vec{E}^*$ is the electric field that exists in the frame of reference moving with the plasma</u>.

Equation (I.57) is most meaningfully interpreted in the context of thermodynamics where it is seen that the time derivative is operating on a term that is related to the specific entropy of the gas. The right hand side then displays the sources and sinks of specific entropy. In order to convert to a description in these terms, we need first to discuss one of the most fundamental quantities of thermodynamics, temperature.

I.12 Temperature and Related Concepts

Temperature is not a necessary variable for our description of the macroscopic behavior of a gas. This is because temperature is uniquely related to pressure and density, which we have already included. However, it is often convenient to use temperature in place of pressure or density and also to express results or concepts in the language of thermal physics. Since our approach has been to progress from the microscale to the macroscale, we introduce temperature from the point of view of the kinetic theory of gases, where it is a measure of the total internal energy of a gas.

$$u_{total} = \frac{f}{2}\, nkT \qquad (I.60)$$

Here n is the number density of gas particles, f denotes the number of degrees of freedom of motion that the gas particles possess and k is Boltzmann's constant ($k = 1.38 \times 10^{-23}$ Joule/deg Kelvin).

$$f = f_{translational} + f_{rotational} + f_{vibrational} \qquad (I.61)$$

The number of translational degrees of freedom can be either 1, 2 or 3 depending on whether the thermal motion of the gas particles is constrained to one-dimension, two dimensions, or is unconstrained. Examples of motion constrained to one-dimension are beads on a wire, and, more relevant to us, a gas of charged particles constrained to move parallel to a magnetic field. Similarly a gas of charged particles constrained to move perpendicular to a magnetical is an example of a case for which $f_{translational} = 2$.

The number of rotational degrees of freedom can be three for a gas of non-colinear molecules. A non-colinear molecule must contain three or more atoms. Of course, if the rotation is constrained to one or two axes, the number of degrees of rotational freedom is correspondingly reduced. Diatomic molecules have at most two degrees of rotational freedom, since the moment of inertia about the common axis is too small to allow it to carry rotational energy. A monatomic gas, such as are virtually all space plasmas, has zero rotational degrees of freedom.

Diatomic and polyatomic molecules can have vibrational degrees of freedom There are two degrees for every vibrational mode, because an oscillator possesses both kinetic and potential energy. Vibrational modes are usually excited at temperatures well above room temperature. For example, the effective number of vibrational degrees of freedom of air in the atmosphere is zero.

The internal energy defined by equation (I.13) results purely from translational motion, but for space plasmas the other kinds of motions do not apply. To be strictly correct, however, equation (I.51) for scalar pressures should be written

$$u_{translational} = \frac{3}{2} p$$

Comparison with (I.60) with f = 3 shows that

$$p = n k T \tag{I.62}$$

This is one of the most important relations of the kinetic theory of gases. To convert it fully to our macroscopic variables defined in I 6, we need only use the mean molecular weight of the gas molecules,

$$\bar{m} \equiv \frac{\rho}{n} \tag{I.63}$$

In a two-component electron-ion plasma, (I.63) becomes (subscripts denote ion and electron quantities)

$$\bar{m} = \frac{m_i n_i + m_e n_e}{n_i + n_e} \cong \frac{m_i}{2} \tag{I.64}$$

in which we have utilized $m_i >> m_e$ and $n_i \cong n_e$. Equation (I.62) is then

$$p = (\frac{k}{\bar{m}}) \rho T \tag{I.65}$$

This may also be written in the usual form given in thermodynamics

$$pv^* = RT \tag{I.66}$$

Here $R \equiv k/m^*$ is the gas constant and $v^* \equiv 1/\rho$ is the specific volume, that is the volume occupied by a kilogram of gas. It is usual in thermodynamics to express densities in units of per-unit-mass rather than per-unit-volume and to designate such quantities as "specific" densities. We will indicate specific densities by a superscript asterisk. Thus the specific internal energy density, u^*_{total} is from (I.60) and the definition for R

$$u^*_{total} = \frac{f}{2} RT \tag{I.67}$$

Equation (I.67) completes the set of relations between the thermodynamic and macroscopic variables that is needed in the following. We turn now to the subject of the heat capacity of the gas, by which is meant the amount of heat that must be added to raise the temperature by one degree. Two thermodynamic variables must be specified to determine the state of a gas, i.e., either p and v^*, p and T or v^* and T, since equation (I.66) gives the third variable once two are known. Clearly in connection with determining heat capacity, T should be one of the chosen variables. There are then two specific heat capacities, c_v and c_p corresponding to whether v^* or p, respectively, is chosen as the second variable. In determining the amount of heat required to change the temperature by one degree, the second variable is held fixed. Thus

$$c_v \equiv \left(\frac{\partial Q^*}{\partial T}\right)_{v^*} \quad \text{(I.68)}$$

and

$$c_p \equiv \left(\frac{\partial Q^*}{\partial T}\right)_p \quad \text{(I.69)}$$

where the specific heat density Q^* is related to the other macroscopic variables by the first law of thermodynamics

$$du^*_{total} = dQ^* - pdv^* \quad \text{(I.70)}$$

The equation says that the internal energy of a fixed amount of gas can be changed by the addition or subtraction of heat to that gas or by work done on or by the gas. To convert (I.70) into a form that can be used when p is chosen as the second variable, substitute for pdv^* from the differential form of equation (I.66)

$$du^*_{total} = dQ^* + v^*dp - RdT \quad \text{(I.71)}$$

Then (I.67), (I.68), and (I.70) taken together yield

$$c_v = \frac{f}{2} R \quad \text{(I.72)}$$

and similarly (I.67), (I.69), and (I.71) yield

$$c_p = \frac{f}{2} R + R = c_v + R \quad \text{(I.73)}$$

The ratio of specific heats, c_p/c_v, is a quantity that appears so frequently it is given its own symbol

$$\gamma \equiv c_p/c_v \quad \text{(I.74)}$$

From (I.72) and (I.73), it is readily seen that

$$\gamma \equiv 1 + \frac{2}{f} \quad \text{(I.75)}$$

The following is a list of examples of values of γ.

f	γ	Example
1	3	motion constrained ‖ to B
2	2	motion constrained ⊥ to B
3	$\frac{5}{3}$	isotropic space plasma
5	$\frac{7}{5}$	air (Earth)
∞	1	many modes of thermal motion

The last thermodynamic quantity with which we desire to make contact is the specific entropy, defined by

$$ds^* \equiv \frac{dQ^*}{T} \qquad (I.76)$$

It proves to be most profitable to express (I.76) in terms of the variables p and v^*. For this first convert to the variable T and v^* through (I.67) and (I.70), with (I.72) and (I.66)

$$ds^* = c_v \frac{dT}{T} + R \frac{dv^*}{v^*} \qquad (I.77)$$

Then eliminate $\frac{dT}{T}$ with the logarithmic differential form of (I.66)

$$\frac{dp}{p} + \frac{dv^*}{v^*} = \frac{dT}{T} \qquad (I.78)$$

which when substituted into (I.77) and combined with (I.73) yields

$$ds^* = c_v \frac{dp}{p} + c_p \frac{dv^*}{v^*} \qquad (I.79)$$

This can be rewritten as

$$ds^* = c_v \, d \, \ell n \frac{p}{\rho^\gamma} \qquad (I.80)$$

where we have used the definitions of v^* and γ.

I.13 Return to the Prognostic Equation for Scalar Pressure

As promised in I.11, we will now interpret equation (I.57) as a statement regarding the behavior of specific entropy. Since for our gas $\gamma = 5/3$, the combination p/ρ^γ can be eliminated between (I.57) and (I.80) resulting in

$$\frac{ds^*}{dt} = \left(\frac{k}{m}\right) \frac{-\nabla\cdot\vec{q} + \vec{j}\cdot\vec{E}^*}{p} \qquad (I.81)$$

where we have used (I.72) and the definition of R. Equation (I.81) makes it clear that (I.57) is a prognostic equation for specific entropy. If $\nabla\cdot\vec{q} = 0$ and $\vec{j}\cdot\vec{E}^* = 0$, s^* is a constant of the motion. Such a situation is referred to as an <u>adiabatic flow</u>, or <u>constant entropy flow</u>. The specific entropy will change as one moves with the flow only if there is a non-zero divergence of the heat flux or if there is <u>Joule dissipation</u> in the frame of reference proving with the plasma, i.e. if $\vec{j}\cdot\vec{E}^* \neq 0$. Note that $\frac{ds^*}{dt}$ can be zero even if $\vec{j}\cdot\vec{E} \neq 0$.

I.14 Adiabatic, Isentropic and Polytropic Flows

In an adiabatic flow ${}^{ds^*}/_{dt} = 0$, and from equation (I.80), this implies

$$p = \alpha\rho^{\gamma} \qquad \text{(I.82)}$$

where α is a constant of the motion. Note that it is not necessary for two different fluid parcels or fluid elements to have the same value of α, but it is necessary for any given fluid parcel to retain the same value of α as the parcel moves with the flow. To appreciate the special status of α, consider (I.82) to be its defining equation. Then (I.80) shows that α is a function of only the specific entropy. Consequently, for an adiabatic flow ${}^{d\alpha}/dt = 0$. Consider the case of a steady state, adiabatic flow. The equation for α becomes

$$\vec{v}\cdot\nabla\alpha = 0 \qquad \text{(I.83)}$$

This is the equation for a streamline constant. Thus α may vary from one streamline to the next, but it has the same value on any given streamline. This means that if α is specified on a two-dimensional surface through which all of the streamlines pass, it is specified throughout the flow.

If it is true that α is the same for all of the fluid elements comprising the fluid, then the flow is said to be isentropic. In this case α is a constant in space and time. An isentropic flow is fully determined by the continuity equation (I.18), the momentum equation (I.23) and the isentropic relation (I.82). An adiabatic flow is also described by these equations, but an additional prescription must be given to specify α.

The enormous simplification that results when the energy equation (I.31) is replaced by (I.82) has lead to the generalization of (I.82) in the form

$$p = \alpha\rho^{n} \qquad \text{(I.84)}$$

in which α and n are specified constants. The exponent, n, is called the polytropic index.

The polytropic assumption is an artifice to render the fluid equations more tractable. The value of n is chosen to produce the best simulation of the thermal condition of the gas possible within the polytropic assumption. For example, if the heat conductivity is very high, heat conduction will keep the gas at a nearly uniform temperature, even though that common temperature may change in time. One can see by comparing equations (I.65) and (I.84) that in order for T to remain uniform while p and ρ are allowed to vary in space, we must have n = 1. The following is a list of commonly used values of n

n	simulation
0	isobaric - constant pressure
1	isothermal - constant temperature - high heat conduction
γ	isentropic
∞	isometric - constant density

There is a more basic way of formulating the simulation that the polytropic assumption performs. The polytropic equation (I.84) is exact for steady state problems (i.e. $\frac{\partial}{\partial t} = 0$) in which the heat flux is proportional to the convected flux of internal energy, $u\vec{V}$, that is

$$\vec{q} = \kappa u\vec{V} \tag{I.85}$$

where κ is a constant, and $\vec{j}\cdot\vec{E}^* = 0$. Then n and κ are related by

$$n = (\tfrac{5}{3} + \kappa)/(1 + \kappa) \tag{I.86}$$

Note that if $\kappa = 0$, i.e. $\vec{q} = 0$, $\gamma = 5/3$ and if $\kappa \to \infty$, $\gamma \to 1$. Thus the adiabatic and isothermal limits are recovered properly.

I.15 The Bernoulli Equation

A special class of solutions to the momentum equation (I.23) exists in the case of steady state, polytropic flows. Rewrite the equation for the case of a scalar pressure and ignore the electrostatic term. We have retained this term for completeness but it is virtually always negligible compared to the other terms, because a plasma tends to maintain itself electrically neutral to a high degree of precision, (§II.2). Then

$$\frac{\partial\vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\ \vec{V} + \frac{\nabla p}{\rho} = \frac{\vec{J}\times\vec{B}}{\rho} - \nabla\phi \tag{I.87}$$

The gravitational force has been replaced by $-\nabla\phi$ and the convective derivative of the velocity has been written out in terms of separate intrinsic and convective terms. With the polytropic relation (I.86), we may write

$$\frac{\nabla p}{\rho} = \nabla\ h^* \tag{I.88}$$

$$\text{where } h^* = \frac{n}{n-1}\ \frac{p}{\rho} \tag{I.89}$$

It is readily verified by the definitions and relations given in I.12 that h^* is the <u>specific enthalpy</u>, that is, $(u+p)/\rho$ where u is defined as if the polytropic index n were the actual ratio of specific heats. Expand $(\vec{V}\cdot\nabla/\vec{V})$ by use of the vector identity

$$(\vec{A}\cdot\nabla)\vec{A} = \nabla\frac{A^2}{2} - \vec{A}\times(\nabla\times\vec{A}) \tag{I.90}$$

where $\vec{A}$ is an arbitrary vector. The scalar product between (I.87) and the velocity vector $\vec{V}$ then becomes

$$\vec{V}\cdot\nabla\left(\frac{V^2}{2} + h^* + \phi\right) = \frac{\vec{J}\cdot(\vec{B}\times\vec{V})}{\rho} \tag{I.91}$$

where the vector identity $(\vec{J}\times\vec{B})\cdot\vec{V} = \vec{J}\cdot(\vec{B}\times\vec{V})$ has been used on the right hand side.

In non-magnetized flows (e.g. in ordinary fluid dynamics) and in flows for which $\vec{B}\times\vec{V} = 0$ everywhere, the right hand side of (I.91) is identically zero. That MHD situations exist for which $\vec{B}||\vec{V}$ everywhere will be demonstrated in the next section. With the right hand side of (I.91) set equal to zero the equation expresses the constancy of the quantity $V^2/2 + h^* + \phi$ along streamlines of the flow. Thus we arrive at the result referred to as the Bernoulli equation

$$\frac{V^2}{2} + h^* + \phi = W \tag{I.92}$$

where W is a streamline constant.

I.16 Divergence of the Anisotropic Pressure Tensor

In situations where it is desired to retain in the Euler equation the anisotropic form of the pressure tensor as given by (I.27), it is useful to reduce the divergence of the tensor to vector operations on scalars and vectors. For this write the pressure tensor in component form as

$$P_{ij} = (p_{\parallel} - p_{\perp})\frac{B_i B_j}{B^2} + p_{\perp}\,\delta_{ij} \tag{I.93}$$

The divergence of P_{ij} is a vector the i-th component of which is

$$\frac{\partial P_{ij}}{\partial x_j} = \frac{B_i B_j}{B^2}\frac{\partial}{\partial x_j}(p_{\parallel} - p_{\perp}) + \frac{\partial p_{\perp}}{\partial x_i} + (p_{\parallel} - p_{\perp})\frac{\partial}{\partial x_j}\left(\frac{B_i B_j}{B^2}\right) \tag{I.94}$$

The component of this vector parallel to $\vec{B}$ is found from

$$B_i\frac{\partial P_{ij}}{\partial x_j} = B_j\frac{\partial p_{\parallel}}{\partial x_j} + (p_{\parallel} - p_{\perp})\,B_i\frac{\partial}{\partial x_j}\left(\frac{B_i B_j}{B^2}\right) \tag{I.95}$$

It is readily verified that

$$B_i\frac{\partial}{\partial x_j}\left(\frac{B_i B_j}{B^2}\right) = -B_i\frac{B_j}{B^2}\frac{\partial B_i}{\partial x_j} \tag{I.96}$$

and hence

$$B_i \frac{\partial P_{ij}}{\partial x_j} = B_i \left[\frac{\partial p_{\parallel}}{\partial x_i} - (p_{\parallel}-p_{\perp}) \frac{B_j}{B^2} \frac{\partial B_i}{\partial x_j}\right] \tag{I.97}$$

In vector notation, this is

$$(\nabla\cdot\overleftrightarrow{P})_{\parallel} = \nabla_{\parallel}p_{\parallel} - (p_{\parallel}-p_{\perp}) \left[\frac{(\vec{B}\cdot\nabla)\vec{B}}{B^2}\right] \tag{I.98}$$

in which the $\parallel$ signs on the vectors denote the component parallel to $\vec{B}$.

To determine the components of (I.94) in the plane perpendicular to $\vec{B}$, Let $\vec{A}$ be any vector in that plane. Then

$$A_i \frac{\partial P_{ij}}{\partial x_j} = A_i \frac{\partial p_{\perp}}{\partial x_i} + (p_{\parallel}-p_{\perp}) A_i \frac{\partial}{\partial x_j} \left(\frac{B_i B_j}{B^2}\right) \tag{I.99}$$

Equation (I.96) holds also when the initial B_i on each side is replaced by A_i. Thus (I.99) becomes

$$A_i \frac{\partial P_{ij}}{\partial x_j} = A_i \left[\frac{\partial p_{\perp}}{\partial x_i} + (p_{\parallel}-p_{\perp}) \frac{B_j}{B^2} \frac{\partial B_i}{\partial x_j}\right] \tag{I.100}$$

or in vector notation

$$(\nabla\cdot\overleftrightarrow{P})_{\perp} = \nabla_{\perp} p_{\perp} + (p_{\parallel}-p_{\perp}) \left[\frac{(\vec{B}\cdot\nabla)\vec{B}}{B^2}\right]_{\perp} \tag{I.101}$$

where the $\perp$ signs on the vectors denote the componenent perpendicular to $\vec{B}$.

I.17 Single Particle Drifts and the Euler Equation

It is usual to begin a course on plasma physics with a derivation of the motion that a point particle of mass m and charge q executes in certain simple electric and magnetic field configurations. The force governing the motion is the Lorentz force

$$\vec{F}_L = q\,\vec{E} + q\,\vec{v} \times \vec{B} \tag{I.102}$$

The second term on the right hand side, the magnetic Lorentz force, always acts perpendicular to the velocity vector. In the absence of an electric field and in a uniform magnetic field, this force causes the particle to move in a circular loop in a plane perpendicular to the magnetic field. The radius r_g of the circle (the gyroradius, r_g) and the angular frequency at which the particle goes around the loop (the gyro-

<u>frequency</u>, $\vec{\omega}_g$) can be be found by balancing the centrifugal force of the motion against the magnetic Lorentz force. Let $\vec{v}_\perp$ be the component of the velocity in the plane perpendicular to $\vec{B}$ (the <u>gyrovelocity</u>). Then the balance of forces is

$$m\vec{\omega}_g \times \vec{v}_\perp = q\vec{v}_\perp \times \vec{B} \tag{I.103}$$

from which we see that

$$\vec{\omega}_g = -\frac{q\vec{B}}{m} \tag{I.104}$$

and from $v_\perp = r_g\omega_g$, we find

$$r_g = \frac{mv_\perp}{qB} \tag{I.105}$$

A charged particle in circular motion generates a magnetic dipole, the magnetic moment of which is the product of the current that the circulating charge produces and the area of the circle

$$\vec{\mu} = (\pi r_g^2)(q\frac{\omega_g}{2\pi}) = -\frac{\frac{1}{2}mv_\perp^2}{B}\frac{\vec{B}}{B} = -\mu\hat{b} \tag{I.106}$$

where

$$\mu \equiv \frac{\frac{1}{2}mv_\perp^2}{B} \tag{I.107}$$

is the magnitude of the dipole moment.

After the treatment of the motion of a particle in a uniform magnetic field, one considers the case of motion in a non-uniform magnetic field and in the presence of an electric field. One of the first non-trivial results of plasma physics is the demonstration that if the spatial and temporal scales of the gyromotion that a particle would execute around the local magnetic field are much less than those of the magnetic and electric fields themselves, μ is a constant of the motion. In this context μ is sometimes called the <u>first adiabatic invariant</u>.

The constancy of μ can be thought of as an example of <u>Lenz's law</u> of electromagnetism, which states that electrical circuits change their currents to counteract externally caused changes in linked magnetic fluxes. In our case, μ is directly proportional to the magnetic flux Φ_g linked by the gyrocircle. Explicitly

$$\mu = \frac{q^2}{2\pi m}\Phi_g \tag{I.108}$$

The constancy of μ is a result of the constancy of Φ_g. Let EMF be the electromotive force per unit charge around the gyrocircle induced by either an intrinsic change in $\vec{B}$ or by a movement of the gyrocircle through a spatially varying magnetic field. Then the particle changes its energy at the rate

$$\frac{d}{dt}\left(\frac{1}{2} m v_\perp^2\right) = q \, EMF \, \frac{\omega_g}{2\pi} \tag{I.109}$$

But

$$EMF = \pi r_g^2 \frac{dB}{dt} \tag{I.110}$$

Substitution of (I.110) into (I.109) results after a few steps in

$$\frac{d}{dt} \ln\left(\frac{\frac{1}{2} m v_\perp^2}{B}\right) = 0 \tag{I.111}$$

The fact that the gyromotion produces a magnetic dipole allows the magnetic Lorentz force acting on the gyro-component of velocity to be replaced by the magnetic force acting on a magnetic dipole. The magnetic force on a field aligned dipole is given in general by

$$\vec{F}_\mu = - \mu \nabla B \tag{I.112}$$

The minus sign occurs because $\vec{\mu}$ is antiparallel to $\vec{B}$ for gyrating charged particles. The original problem of determining the motion of a charged particle with velocity components $\vec{v}_\perp$ perpendicular to $\vec{B}$ and $v_\parallel$ parallel to $\vec{B}$ becomes replaced by the problem of determining the motion of a current loop of mass m, charge q and magnetic moment $\vec{\mu}$ which has velocity components $\vec{V}_D$ perpendicular to $\vec{B}$ and $v_\parallel$ parallel to $\vec{B}$.

The motion of the current loop perpendicular to the field as given by its drift velocity, $\vec{V}_D$, can be different for ions and electrons and, therefore, can result in electrical currents, called drift currents. The purpose of this section is to show that when all of the drift currents are combined, one recovers the Euler equation.

The drift velocity is obtained by considering the forces acting on the charged loop in the plane perpendicular to $\vec{B}$

$$\vec{F}_\perp = q\vec{E}_\perp + q(\vec{V}_D \times \vec{B}) - \mu \nabla_\perp B - m v_\parallel^2 \left[\left(\frac{\vec{B}}{B} \cdot \nabla\right) \frac{\vec{B}}{B}\right]_\perp + m\vec{g}_\perp \tag{I.113}$$
$$= m \frac{d\vec{V}_D}{dt}$$

The fourth term on the right hand side is the centrifugal force that results from the motion of the loop along a curved line of force. The term in brackets is the curvature of the field line. The other terms require no further explanation. Solve (I.113) for $\vec{V}_D$ to find

$$\vec{V}_D = \frac{\vec{E}\times\vec{B}}{B^2} + \frac{\mu}{qB}\frac{\vec{B}}{B} \times \nabla B + \frac{mv_{\parallel}^2}{qB}\frac{\vec{B}}{B} \times [(\frac{\vec{B}}{B}\cdot\nabla)\frac{\vec{B}}{B}]$$

$$+ \frac{m}{qB}\frac{\vec{B}}{B} \times \frac{d\vec{V}_D}{dt} + \frac{m}{qB}\vec{g} \times \frac{\vec{B}}{B} \qquad (I.114)$$

The subscript $\perp$ is now superfluous and has been dropped.

The first term on the right hand side of (I.114) is the <u>electric field drift</u>, the second term is the <u>gradient drift</u>, the third term is the <u>curvature drift</u>, the fourth term is the <u>inertial drift</u> and the last term is the <u>gravity drift</u>. The inertial drift is sometimes written in terms of the electric field, in which form it is called the polarization drift. The replacement is possible because in drift motion calculations, all quantities except $\vec{V}_D$ and $\vec{E}$ are assumed to be constant. Then by differentiating (I.114) with respect to time and dropping the second time derivative of $\vec{V}_D$, we find

$$\frac{d\vec{V}_D}{dt} = \frac{\partial\vec{E}}{\partial t} \times \frac{\vec{B}}{B^2} \qquad (I.115)$$

In this way the inertial drift term becomes the expression for the <u>polarization drift</u>

$$\frac{m}{qB}\frac{\vec{B}}{B} \times \frac{d\vec{V}_D}{dt} = \frac{m}{qB^2}\frac{\partial\vec{E}}{\partial t} \qquad (I.116)$$

If now the drift velocities of the ions and electrons are combined to form an expression for the drift current

$$\vec{J}_D = e[n_i(\vec{V}_D)_i - n_e(\vec{V}_D)_e] \qquad (I.117)$$

one finds by inspection of (I.114) (with $q_i = e$ and $q_e = -e$)

$$\vec{J}_D = \rho_c \frac{\vec{E}\times\vec{B}}{B^2} + \frac{p_\perp}{B^2}\frac{\vec{B}}{B} \times \nabla B$$

$$+ \frac{p_{\parallel}}{B}\frac{\vec{B}}{B} \times [(\frac{\vec{B}}{B}\cdot\nabla)\frac{\vec{B}}{B}] + \frac{\rho}{B}\frac{\vec{B}}{B} \times \frac{d\vec{V}_D}{dt}$$

$$+ \frac{\rho}{B}\vec{g} \times \frac{\vec{B}}{B} \qquad (I.118)$$

In this result the values of $p_\perp$ and $p_{\parallel}$ have been constructed out of the single particle parameters in a manner that is consistent with their definitions in (1.12).

$$p_\perp = \frac{1}{2} n_i m_i (v_\perp^2)_i + \frac{1}{2} n_e m_e (v_\perp^2)_e \tag{I.119}$$

$$p_\parallel = n_i m_i (v_\parallel^2)_i + n_e m_i (v_\parallel^2)_e \tag{I.120}$$

The factor ½ appears in (I.119) and not in (I.120) because $v_\perp^2 = v_x^2 + v_y^2$ embodies two components of motion.

To obtain the total current flowing in the plasma that has now been created out of the bringing together of separate ion and electron components, the magnetization current $\vec{J}_M$ must be added, which arises whenever there is an inhomogeneous distribution of magnetic dipoles.

$$\vec{J} \equiv \vec{J}_{Total} = \vec{J}_D + \vec{J}_M \tag{I.121}$$

where $\vec{J}_M$ is given explicitly in terms of dipole moment distribution by

$$\vec{J}_M = \nabla \times (n_i \vec{\mu}_i + n_e \vec{\mu}_e) \tag{I.122}$$

This reduces immediately to

$$\vec{J}_M = -\nabla \times \left(\frac{p_\perp}{B} \frac{\vec{B}}{B}\right) \tag{I.123}$$

$$= \frac{\vec{B}}{B^2} \times \nabla p_\perp - 2 \frac{p_\perp}{B^2} \frac{\vec{B}}{B} \times \nabla B - \frac{p_\perp}{B^2} \nabla \times \vec{B} \tag{I.124}$$

The addition of (I.118) and (I.124) gives

$$\vec{J} = \rho_c \frac{\vec{E}\times\vec{B}}{B^2} + \frac{\vec{B}}{B^2} \times \nabla p_\perp + \frac{p_\parallel}{B} \frac{\vec{B}}{B} \times \left[\left(\frac{\vec{B}}{B}\cdot\nabla\right)\frac{\vec{B}}{B}\right]$$

$$- p_\perp \left(\frac{\vec{B} \times \nabla B}{B^3} + \frac{\nabla\times\vec{B}}{B^2}\right) + \frac{\rho}{B} \frac{\vec{B}}{B} \times \frac{d\vec{V}_D}{dt} + \frac{\rho}{B} \vec{g} \times \frac{\vec{B}}{B} \tag{I.125}$$

Two further algebraic reductions are needed to reach the desired form. These are the two vector identities

$$\frac{\vec{B}}{B} \times \left[\left(\frac{\vec{B}}{B}\cdot\nabla\right)\frac{\vec{B}}{B}\right] = \frac{B}{B} \times \left[\frac{(\vec{B}\cdot\nabla)\vec{B}}{B^2}\right] \tag{I.126}$$

and

$$\frac{\vec{B}\times\nabla B}{B^3} + \frac{\nabla\times\vec{B}}{B^2} = \frac{\vec{B}}{B^2} \times \left[\frac{(\vec{B}\cdot\nabla)\vec{B}}{B^2}\right] \tag{I.127}$$

(The validity of the second equation requires $\vec{J}\perp\vec{B}$, which is appropriate to this discussion). The identities allow (I.125) to be rewritten as

$$\vec{J} = \frac{\vec{B}}{B^2} \times \left[\rho \frac{d\vec{V}_D}{dt} + \nabla p_\perp + (p_\parallel - p_\perp) \frac{(\vec{B}\cdot\nabla)\vec{B}}{B^2} - \rho_c \vec{E} - \rho\vec{g} \right] \quad (I.128)$$

Equation (I.128) can now be inverted to solve for $\rho \frac{d\vec{V}_D}{dt}$, with the understanding that the result applies in the plane perpendicular to $\vec{B}$.

$$\rho \frac{d\vec{V}_D}{dt} = -\nabla p_\perp - (p_\parallel - p_\perp) \frac{(\vec{B}\cdot\nabla)\vec{B}}{B^2} + \rho_c \vec{E} + \vec{J}\times\vec{B} + \rho\vec{g} \quad (I.129)$$

The final expression, equation (I.129), is seen to be identical with the perpendicular component of the Euler equation (I.23) in which $\vec{V}_D$ is identified with $\vec{V}$ and the form of the divergence of the pressure tensor is given by (I.101). It should be noted that while the result demonstrates that the microscopic and macroscopic descriptions are formally identical, the use of particle drift theory is restricted to situations in which the magnetic field is presumed known. In the MHD description, the magnetic field is one of the dependent variables, and thus situations can be treated in which the magnetic field is in part or completely determined by the plasma.

I.18 Limitations to the Use of the Macroscopic Equations

As they now stand the macroscopic equations are not a closed set. To obtain a fully complete macroscopic description of a plasma, it is in principle necessary to compute all of the moments of the Boltzmann equation. The truncation at $n = 2$ necessarily leaves more dependent variables than equations. There is no equation for the highest order dependent variable, the heat flux vector, $\vec{q}$. The problems that can be treated either set $\vec{q}$ equal to zero (adiabatic flows), or use the artifice of a polytropic index to simulate the effect of heat flux (polytropic flows), or introduce an explicit equation for $\vec{q}$, such as a thermal conduction equation. More elaborate forms of equations for $\vec{q}$ are being evolved in connection with the theory of the solar wind.

The components of the pressure tensor also are not completely determined within the derived equations. It was noted that the specification of the viscous-like components required results from kinetic theory to obtain the coefficients of viscosity. The anisotropic form of the pure-pressure terms depends on the validity of the approximation that the pressure is isotropic in the plane perpendicular to the magnetic field. When gradients in the plasma or the fields are comparable to the gyro-radius of the ions, this is not a valid approximation.

There is an even more fundamental limitation to the equations at the stage of the development we have now reached. Consider a steady state ($\partial/\partial t = 0$) problem which has only the minimum number of dependent variables to still qualify as an MHD problem, namely $\vec{V}$, ρ, p and $\vec{B}$. The current density $\vec{J}$ can be expressed in terms of $\vec{B}$ through the Maxwell equation (I.35). There are a total of eight unknowns, if the vector components are taken into account. However, the continuity equation, the Euler equation, the adiabatic relation and the remaining Maxwell equation involving $\vec{B}$ (I.33) total only six, including the three vector components of the Euler equation.

The missing equations are supplied through the computation of the first charge-moment of the Boltzmann equation (I.49 with n = 1). This results in an equation called the <u>generalized Ohm's law</u> that relates the electric field vector $\vec{E}$ to the other dependent variables. Then the Maxwell equation (I.34) provides three additional equations. (At the same time Maxwell equation (I.33) becomes redundant because it follows from (I.34) with the prescription that $\nabla\cdot\vec{B} = 0$ at some initial instant.)

The role that the generalized Ohm's law and its approximate form the hydromagnetic approximation play in MHD is sufficiently important to be considered in a separate major section.

II. THE HYDROMAGNETIC APPROXIMATION AND ITS CONSEQUENCES

II.1 The Generalized Ohm's Law

The first charge moment of the Boltzmann equation (I.49) leads after a fairly lengthy series of intermediate steps (e.g. Rossi and Olbert, 1970) to the following expression in which no approximations have been made other than replacing the collision integral by an effective collision time.

$$\frac{\partial J_i}{\partial t} + \frac{e}{m_a}\frac{\partial}{\partial x_j}(P_{ij})_a - \frac{e}{m_b}\frac{\partial}{\partial x_j}(P_{ij})_b + \frac{\partial}{\partial x_j}(J_iV_j + J_jV_i - \rho_c V_i V_j)$$
$$= \alpha\{\vec{E} + [\vec{V} + \frac{e}{\alpha}(\frac{1}{m_a} - \frac{1}{m_b})(\vec{J} - \rho_c\vec{V})] \times \vec{B}\}_i + \rho_c\vec{g} - (\vec{J} - \rho_c\vec{V})/\tau \qquad \text{(II.1)}$$

in which subscripts <u>a</u> and <u>b</u> signify ions and electrons, respectively. The ions are assumed to be singly charged and all to have the same mass, m_a. The parameter α is defined by

$$\alpha \equiv \frac{e^2}{m_a m_b}\rho + e(\frac{1}{m_a} - \frac{1}{m_b})\rho_c \qquad \text{(II.2)}$$

The quantity τ that enters into (II.1) is the time scale for momentum exchange by means of collisions between the ion gas and electron gas that together make up the plasma. The time scale for coulomb collisions is given by Spitzer (1956).