variables to still qualify as an MHD problem, namely $\vec{V}$, ρ, p and $\vec{B}$. The current density $\vec{J}$ can be expressed in terms of $\vec{B}$ through the Maxwell equation (I.35). There are a total of eight unknowns, if the vector components are taken into account. However, the continuity equation, the Euler equation, the adiabatic relation and the remaining Maxwell equation involving $\vec{B}$ (I.33) total only six, including the three vector components of the Euler equation.

The missing equations are supplied through the computation of the first charge-moment of the Boltzmann equation (I.49 with n = 1). This results in an equation called the generalized Ohm's law that relates the electric field vector $\vec{E}$ to the other dependent variables. Then the Maxwell equation (I.34) provides three additional equations. (At the same time Maxwell equation (I.33) becomes redundant because it follows from (I.34) with the prescription that $\nabla\cdot\vec{B} = 0$ at some initial instant.)

The role that the generalized Ohm's law and its approximate form the hydromagnetic approximation play in MHD is sufficiently important to be considered in a separate major section.

II. THE HYDROMAGNETIC APPROXIMATION AND ITS CONSEQUENCES

II.1 The Generalized Ohm's Law

The first charge moment of the Boltzmann equation (I.49) leads after a fairly lengthy series of intermediate steps (e.g. Rossi and Olbert, 1970) to the following expression in which no approximations have been made other than replacing the collision integral by an effective collision time.

$$\frac{\partial J_i}{\partial t} + \frac{e}{m_a}\frac{\partial}{\partial x_j}(P_{ij})_a - \frac{e}{m_b}\frac{\partial}{\partial x_j}(P_{ij})_b + \frac{\partial}{\partial x_j}(J_iV_j + J_jV_i - \rho_c V_i V_j)$$

$$= \alpha\{\vec{E} + [\vec{V} + \frac{e}{\alpha}(\frac{1}{m_a} - \frac{1}{m_b})(\vec{J}-\rho_c\vec{V})] \times \vec{B}\}_i + \rho_c\vec{g} - (\vec{J}-\rho_c\vec{V})/_\tau \qquad \text{(II.1)}$$

in which subscripts a and b signify ions and electrons, respectively. The ions are assumed to be singly charged and all to have the same mass, m_a. The parameter α is defined by

$$\alpha \equiv \frac{e^2}{m_a m_b}\rho + e(\frac{1}{m_a} - \frac{1}{m_b})\rho_c \qquad \text{(II.2)}$$

The quantity τ that enters into (II.1) is the time scale for momentum exchange by means of collisions between the ion gas and electron gas that together make up the plasma. The time scale for coulomb collisions is given by Spitzer (1956).

To transform (II.1) into a more useful expression, we make two highly accurate approximations, the first of which is obvious, and the second will be justified in Section II.2.

$$\frac{1}{m_a} << \frac{1}{m_b} \tag{II.3}$$

$$\rho_c \vec{V} << \vec{J} \tag{II.4}$$

All terms containing the left hand sides of (II.3) and (II.4) will be dropped in (II.1). At this point, we revert to vector notation and to the standard nomenclature in which subscripts i and e denote the ion and electron components of the plasma. Also to simplify nomenclature, let

$$\eta \equiv \frac{1}{\alpha\tau} = \frac{m_e}{e^2 n \tau} \tag{II.5}$$

Then if (II.1) is solved for $\vec{E}$, there results

$$\vec{E} = -\vec{V} \times \vec{B} + \eta\vec{J} - \frac{1}{ne} \nabla\cdot\overleftrightarrow{P}_e + \frac{1}{ne} \vec{J} \times \vec{B} + \frac{m_e}{e^2 n} [\frac{\partial\vec{J}}{\partial t} + \nabla\cdot(\vec{J}\vec{V} + \vec{V}\vec{J})] \tag{II.6}$$

in which an obvious diadic notion has been used.

It can be seen from equation (I.59) that the sum of all of the terms on the right hand side of (II.6) exclusive of the first is precisely $\vec{E}^*$, the electric field in the co-moving frame of reference. Thus (II.6) can be written in the more revealing form

$$E = -\vec{V} \times \vec{B} + \vec{E}^* \tag{II.7}$$

The electric field is the sum of a motional part owing to translation of the plasma in the presence of a magnetic field (or better said, as will be seen later, owing to translation of the plasma carrying a magnetic field) and a static or material part which is the electric field in the plasma itself. $\vec{E}^*$ is given explicitly by

$$\vec{E}^* = \eta\vec{J} - \frac{1}{ne}\nabla\cdot\overleftrightarrow{P}_e + \frac{1}{ne} \vec{J} \times \vec{B} + \frac{m_e}{e^2 n} [\frac{\partial\vec{J}}{\partial t} + \nabla\cdot(\overrightarrow{\overrightarrow{JV}} + \overrightarrow{\overrightarrow{VJ}})] \tag{II.8}$$

The first contribution to E^* is the usual ohmic resistance term. The second term gives the ambipolar electric field that is needed to restrain the electrons when they are driven by a pressure gradient and thereby to maintain the plasma neutral. The third term expresses the Hall effect, the meaning of which will be given below. The final term in brackets represents the effect of electron inertia.

In any intended application of (II.6), some terms will be much smaller than the largest terms and they can therefore be neglected. To determine which are the negligible terms, we perform a scale analysis by comparing the magnitudes of all terms with that of some reference term. Take the first member of the right hand side to be the reference term and proceed to define dimensionless ratios with the remaining terms

$$R_1 \equiv \frac{VB}{\eta J} = \frac{VB}{\frac{m_e}{e^2 n\tau}\,\frac{B}{\mu_o L}} = nVL\tau\,\frac{\mu_o e^2}{m_e} \qquad \text{(II.9)}$$

$$R_2 \equiv \frac{VB}{\frac{1}{ne}\,\frac{P_e}{L}} = \frac{VB}{\frac{k}{ne}\,\frac{nT}{L}} = \frac{e}{k}\,\frac{VBL}{T} \qquad \text{(II.10)}$$

$$R_3 = \frac{VB}{\frac{1}{en}\,\frac{B^2}{\mu_o L}} = \frac{VLn}{B}\,e\mu_o \qquad \text{(II.11)}$$

$$R_4 = \frac{VB}{\frac{m_e}{e^2 n}\,\frac{J}{T}} = L^2 n\,\frac{\mu_o e^2}{m_e} \qquad \text{(II.12)}$$

In these expressions, L and T represent characteristic length and time scales of the flow.

As an example of a typical set of space parameters to evaluate the sizes of these ratios, take $V = 10^5\ ms^{-1}$, $L = 10^6 m$, $n = 10^7 m^{-3}$, $B = 10^{-8} T$, $T_e = 10^5\ {}^oK$, and note that $\mu_o e^2/m_e \cong 3x10^{-14}$, $e/k \cong 10^4$, $e\mu_o \cong 2x10^{-25}$ in MKS units. Then $R_1 = 3x10^4\tau >> 1$, $R_2 = 10^2 >> 1$, $R_3 = 10 > 1$, $R_4 = 3x10^5 >> 1$.

The example shows that in a typical space situation, $-\vec{V}x\vec{B}$ is comfortably larger than all of the other terms on the right hand side, except possibley $\vec{J}x\vec{B}/ne$. That is, generally

$$|\vec{E}^{\,*}| << |\vec{V}x\vec{B}| \qquad \text{(II.13)}$$

Thus in most space applications, the generalized Ohm's law is replaced by the <u>hydromagnetic equation</u>, also known as the <u>hydromagnetic approximation</u> or the <u>hydromagnetic limit</u> of the generalized Ohm's law.

$$\vec{E} = -\vec{V}x\vec{B} \qquad \text{(II.14)}$$

That is, to a good approximation, the electric field may be regarded as a purely motional field. It is important to be aware of the possibility that other terms in (II.6) could dominate in restricted regions of space.

The subject of magnetic merging, to which we return briefly later, is an example where (II.14) is assumed to be violated.

Even in normal space conditions there is a somewhat better approximation for $\vec{E}$ than that given by (II.14). As was seen in the numerical example, the term which tends most seriously to invalidate (II.14) is $\vec{J} \times \vec{B}/en$. If this term is retained, then the resulting expression for $\vec{E}$ will be improved. In our numerical example, the remaining corrections would be one percent or less. Thus a more accurate expression is

$$\vec{E} = -\left(\vec{V} - \frac{\vec{J}}{en}\right) \times \vec{B} \tag{II.15}$$

If we consider the plasma to be composed of an ion gas component moving with velocity $\vec{V}_i$ and an electron gas component moving with velocity $\vec{V}_e$, then with the same two approximations given by (II.3) and (II.4) we may write

$$\rho\vec{V} = m_i n \vec{V}_i \tag{II.16}$$

$$\vec{J} = en(\vec{V}_i - \vec{V}_e) \tag{II.17}$$

where n is the common number density of the two gases (cf. II.4). With these expressions for $\vec{V}$ and $\vec{J}$, (II.15) becomes

$$\vec{E} = -\vec{V}_e \times \vec{B} \tag{II.18}$$

Equation (II.18) shows that in the more accurate version of the hydromagnetic approximation in which the Hall effect term is retained, the electric field is the motional field of the electron gas component of the plasma.

II.2 Charge Neutrality and Related Approximations

The expression (II.6) for the electric field permits a verification of the basic condition of charge neutrality that has already been invoked several times. The relative difference in the number densities of ions and electrons can be parameterized by the dimensionless ratio ρ_c/en, where n is the common (average) number density of the two charge particles. The space charge density ρ_c is related to $\vec{E}$ through the Maxwell equation (I.32). As in the previous section, we perform a scale analysis of this equation with the foreknowledge that the term $-\vec{V} \times \vec{B}$ makes the biggest contribution to $\vec{E}$.

$$\frac{\rho_c}{en} \approx \frac{\varepsilon_o VB}{n eL} \tag{II.19}$$

Next make the replacements $B/L = \mu_o J = \mu_o en|\vec{V}_i - \vec{V}_e|$, $V = V_i$, and $\varepsilon_o\mu_o = 1/c^2$. Then (II.19) becomes

$$\frac{\rho_c}{en} \approx \frac{V_i|\vec{V}_i - \vec{V}_e|}{c^2} \qquad \text{(II.20)}$$

In the non-relativistic plasmas that populate the solar system, charge neutrality is seen to be strongly obeyed.

If instead of VB, we use the largest term in $\vec{E}^*$, namely $\frac{1}{ne}\vec{J}\times\vec{B}$, to represent $\vec{E}$, the dimensionless ratio of number densities becomes

$$\frac{\rho_c}{en} \approx \frac{(\vec{V}_i - \vec{V}_e)^2}{c^2} \qquad \text{(II.21)}$$

A similar analysis can be done for the electrostatic force in the Euler equation (I.23) which was stated earlier to be negligible compared to the ponderomotive force. This claim can be verified directly by a scale analysis

$$\frac{\rho_c E}{JxB} \approx \frac{\varepsilon_o E^2}{LJB} \qquad \text{(II.22)}$$

Make the substitutions $E = VB$, $J = B/\mu_o L$, $\varepsilon_o\mu_o = c^2$ to find

$$\frac{\rho_c E}{JxB} \approx \frac{V^2}{c^2} \qquad \text{(II.23)}$$

This ratio is of the same order of smallness as the space charge density-number density ratio. Scale analysis shows the electrostatic energy density, which enters into Poynting's theorem (I.44) and the equation for the conservation of energy, is smaller than the magnetic energy density by the same velocity ratio

$$\frac{\frac{\varepsilon_o}{2} E^2}{B^2/2\mu_o} \approx \frac{\varepsilon_o\mu_o V^2 B^2}{B^2} = \frac{V^2}{c^2} \qquad \text{(II.24)}$$

Finally, the displacement current $\varepsilon_o\mu_o\, \partial\vec{E}/\partial t$ that appears in the Maxwell equation (I.35) is also negligible compared to $\vec{\nabla} \times \vec{B}$.

$$\frac{\varepsilon_o\mu_o\, \partial\vec{E}/\partial t}{\nabla\times\vec{B}} \approx \frac{\varepsilon_o\mu_o VB/T}{B/L} \qquad \text{(II.25)}$$

With the substitutions $\varepsilon_o \mu_o = {}^1/_{c^2}$ and ${}^L/_T = V$, this becomes

$$\frac{\varepsilon_o \mu_o \, \partial \vec{E}/\partial t}{\nabla \times \vec{B}} \approx \frac{V^2}{c^2} \tag{II.26}$$

II.3 Poynting's Theorem in the Hydromagnetic Limit

In this and subsequent sections the consequences that follow from the hydromagnetic approximation (II.14) are developed. Poynting's theorem (I.44) becomes especially simple under the hydromagnetic approximation and has an easily understood meaning where Equation (II.19) is used to replace the electric field, and the electric energy density term is dropped in accordance with (II.24). There results

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_o}\right) + \nabla\cdot\left(\frac{B^2}{\mu_o}\vec{V}_\perp\right) = -\vec{V}_\perp\cdot(\vec{J}\times\vec{B}) \tag{II.27}$$

in which the Poynting vector appears as

$$\vec{S} = \frac{-(\vec{V}\times\vec{B})\times\vec{B}}{\mu_o} = \frac{B^2}{\mu_o}\left[\vec{V}-(\hat{b}\cdot\vec{V})\hat{b}\right]$$

$$= \frac{B^2}{\mu_o}\vec{V}_\perp \tag{II.28}$$

where $\vec{V}_\perp$ is the component of $\vec{V}$ perpendicular to $\vec{B}$. $\vec{S}$ has the appearance of a flux of magnetic energy density, except that magnetic energy density is $B^2/2\mu_o$. Poynting's vector, $\vec{S}$, is properly interpreted as the magnetic enthalpy flux density. In strict analogy with the mechanical enthalpy flux density $(u+p)\vec{V}$, $\vec{S}$ contains both the energy flux density of the field $(B^2/2\mu_o)\vec{V}$ and the work required to move that energy against the pressure $B^2/2\mu_o$.

The electromechanical energy conversion term $-\vec{E}\cdot\vec{J}$ becomes after an intermediate step $-\vec{V}\cdot(\vec{J}\times\vec{B})$. Since $\vec{V}_\parallel\cdot(\vec{J}\times\vec{B}) = 0$, where $\vec{V}_\parallel = (\hat{b}\cdot\vec{V})\hat{b}$ is the component of $\vec{V}$ parallel to $\vec{B}$, the expression $\vec{V}\cdot(\vec{J}\times\vec{B})$ can be written as $\vec{V}_\perp\cdot(\vec{J}\times\vec{B})$ without approximation. The term is seen to be the power resulting from applying the pondermotive force $(\vec{J}\times\vec{B})$ against the flow velocity $\vec{V}_\perp$. Thus if $(\vec{J}\times\vec{B})$ opposes the motion, the flow is slowed and energy is transformed from mechanical form to magnetic form, and vice versa.

II.4 Equipotential Fieldlines and Streamlines in Steady State Hydromagnetic Flows

An immediately deduced and far reaching property of steady state hydromagnetic flows results from the Maxwell equation (I.34, also known as Faraday's induction law). In steady state, (I.34) becomes

$$\nabla \times \vec{E} = 0 \qquad \text{(II.29)}$$

The electric field is therefore given by the gradient of an electrical potential Φ_E

$$\vec{E} = -\nabla\Phi_E \qquad \text{(II.30)}$$

The vector $\vec{E}$ is everywhere normal to surfaces of constant Φ_E (equipotential surfaces). In the hydromagnetic limit (II.14) both the velocity vector $\vec{V}$ and the magnetic field vector $\vec{B}$ are everywhere orthogonal to $\vec{E}$, and therefore lie in equipotential surfaces. It follows that streamlines of the flow and magnetic field lines are confined to equipotential surfaces and are therefore equipotential lines. This result can be obtained formally by replacing $\vec{E}$ by $-\nabla\Phi_E$ in (II.14) and scalar multiplying the resulting relation first with $\vec{V}$ and then with $\vec{B}$.

$$\vec{V}\cdot\nabla\Phi_E = 0 \qquad \text{(II.31)}$$

$$\vec{B}\cdot\nabla\Phi_E = 0 \qquad \text{(II.32)}$$

It should be kept in mind that some time dependent flows can be converted to steady state flows by an appropriate Galilean transformation. For example it is sometimes useful in problems involving propagating plane waves or planar discontinuities to transform to the frame of reference moving with the wave or the discontinuity to take advantage of (II.31) and (II.32). Solar system examples exist in the form of structures in the solar wind that corotate with the sun and possibly in the form of corotating structures in the magnetospheres of Jupiter and Saturn. The structures would appear stationary in the corresponding corotating frame of reference.

In the case of the sun a further simplification applies that illustrates a special but important class of steady flows. Assume constancy prevails in the frame of reference corotating with the sun. This may be considered an idealization of a circumstance in which solar surface conditions change slowly on a relevant solar wind flow time. Then (II.31) and (II.32) may be used as valid approximations to the actual situation. In the photosphere the solar wind velocity is essentially zero and the electrical resistivity is low. Thus in the photosphere the main contributors to $\vec{E}$ in the generalized Ohm's law can be taken to be negligibly small. This results in the characterization of the photosphere as an equipotential surface in the corotating frame. (It would not be so characterized in the inertial frame since the rotational motion of the photospheric plasma gives a $-\vec{V}_{rotation} \times \vec{B}_{photosphere}$ contribution to $\vec{E}$ in that frame.) Now in the corotating reference frame all of space filled by the solar wind is linked to the photosphere by equipotential field (and flow) lines.

It follows that since the photosphere is an equipotential surface, Φ_E = constant in the region of space filled by the solar wind. Equation (II.30) then gives $\vec{E} = 0$ everywhere in the solar wind in the corotating reference frame. The hydromagnetic equation (II.14) allows the further deduction that $\vec{V} || \vec{B}$ everywhere in the solar wind in the corotating reference frame. It will be recalled from Section (I.15) that this condition permits the construction of a Bernoulli integral for the flow in the corotating reference frame.

The situation just described for the solar wind can be generalized to a statement about a special class of steady flows. If the flow streamlines or the magnetic field lines comprising a continuous flow or field domain pass through an equipotential surface anywhere within the domain, the entire domain is an equipotential volume, the electric field within it is zero and the streamlines and flowlines within it coincide.

A further useful relation between the magnitudes of the velocity and magnetic field can be derived for these equipotential domain flows. Since it is usually necessary to make a Galilean transformation from the reference frame in which the velocity $\vec{V}$ is given and the reference frame of the equipotential domain, let $\vec{V}^*$ be the required transformation velocity. The flow velocity in the reference frame of the equipotential domain is then $(\vec{V}-\vec{V}^*)$. In the solar wind problem in which $\vec{V}$ is given in the inertial reference frame, $\vec{V}^*$ is the corotation velocity $\vec{\Omega} \times \vec{r}$, where $\vec{\Omega}$ is the solar angular velocity vector and $\vec{r}$ is the radius vector in a heliocentric spherical polar coordinate system. The condition $\vec{B} || (\vec{V}-\vec{V}^*)$ can be expressed as

$$\vec{B} = \kappa\rho(\vec{V}-\vec{V}^*) \tag{II.33}$$

where κ is an as yet unknown scalar function of space and ρ is mass density. The reason for including ρ explicitly as part of the coefficient to $(\vec{V}-\vec{V}^*)$ is to take advantage of the continuity equation (I.18), which in steady state is

$$\nabla\cdot[\rho(\vec{V}-\vec{V}^*)] = 0 \tag{II.34}$$

The divergence of (II.33) is zero by Maxwell's equation (I.33). Thus with (II.34) there results

$$\rho(\vec{V}-\vec{V}^*)\cdot\nabla\kappa = 0 \tag{II.35}$$

This is the equation of a streamline constant of the flow in the reference frame of the equipotential domain, that is, κ is constant on the streamlines of $(\vec{V}-\vec{V}^*)$. If κ is constant across any surface linked by all of the streamlines in the equipotential domain, then κ is a constant throughout the domain.

II.5 Freezing Laws

In 1942, Alfvén showed that if a fluid moves such that its velocity is related to the electric and magnetic fields in accordance with equation (II.14), then the Faraday induction law (eq. I.34) imposes a constraint on the motion that can be described as a freezing of the fluid to the magnetic field, or in Alfvén's words "the matter of the liquid is fastened to the lines of force" (Alfvén, 1942). If we substitute for the electric field from eq. (II.14) into (I.34), we obtain:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{V} \times \vec{B}) \qquad \text{(II.36)}$$

The freezing law, which follows simply as a kinematical consequence of (II.36), states that the magnetic flux through a closed loop that moves with the fluid is constant in time. The demonstration of this result proceeds as follows.

The magnetic flux through a closed loop, ℓ, is defined by

$$F \equiv \int \vec{B}\cdot\hat{n}\, da \qquad \text{(II.37)}$$

where da is an element of area on any surface which has ℓ as its perimeter. Gauss' theorem and the divergence-free condition on $\vec{B}$ (eq. 1.33) guarantee that the magnetic flux is the same through all surfaces sharing a common perimeter. The freezing law can be expressed mathematically by

$$\frac{dF}{dt} = 0 \qquad \text{(II.38)}$$

The symbol for the total time derivative is used to indicate that F is to be evaluated in reference to a linked set of fluid elements that move with the fluid. Equation (I.20) for the total derivative is not appropriate, however, because F is not a locally defined quantity. We must evaluate (II.38) using the integral form of F, eq. (II.37) explicitly.

Refer to Figure (II.1) which shows a closed loop of fluid elements, ℓ, at two successive instants, t and $t + \Delta t$. An enclosed volume is formed by the two surfaces S_1 and S_2, that have $\ell(t)$ and $\ell(t + \Delta t)$ as their perimeters, and the generalized cylinder, S_3 generated by the motion of ℓ. Let F be the magnetic flux enclosed by ℓ, and denote by subscripts 1, 2 and 3 the fluxes through the surfaces, S_1, S_2 and S_3. Then if the normal vectors to S_1 and S_2 are chosen to lie on the same side of each surface relative to the flow, as indicated,

$$\frac{dF}{dt} = \frac{F_2(t + \Delta t) - F_1(t)}{\Delta t} \qquad \text{(II.39)}$$

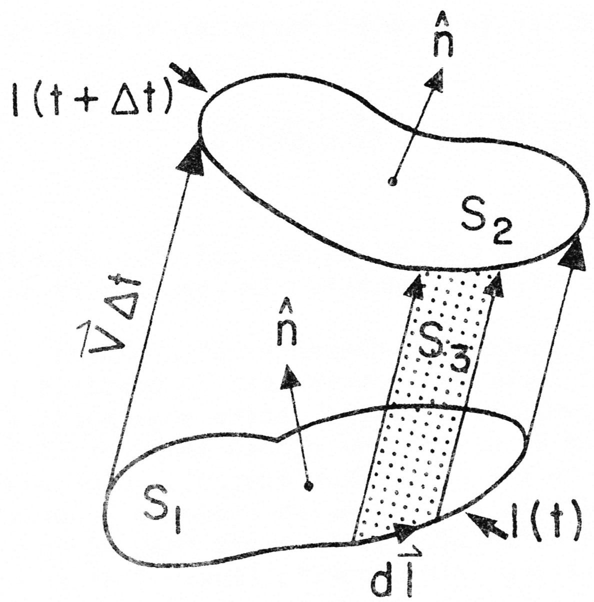

Figure II.1 Generalized cyclinder formed by the motion of closed line "frozen" to the fluid.

where the limit $\Delta t \rightarrow 0$ is understood. The divergence-free condition on $\vec{B}$ requires a zero net flux through the three surfaces at any time. In particular

$$-F_1(t + \Delta t) + F_2(t + \Delta t) + F_3 = 0 \qquad \text{(II.40)}$$

In this equation it is recognized that the outward pointing normal to S_1 is needed to utilize Gauss' theorem, and hence, the change in the sign on F_1 relative to F_2 and F_3.

Now eliminate $F_2(t + \Delta t)$ between eqs. (II.39) and (II.40) and replace the fluxes by their integral forms, (I.27)

$$\frac{dF}{dt} = \frac{1}{\Delta t}\left[\int_{S_1} [\vec{B}(t + \Delta t) - B(t)]\cdot\hat{n}da - \int_{S_3} \vec{B}\cdot\hat{n}da\right] \qquad \text{(II.41)}$$

The first integral is evidently $\int_{S_1} \frac{\partial B}{\partial t}\cdot\hat{n}da$. The second term on the right hand side can also be converted into an integral over the surface S_1 with the following identities

$$\int_{S_3} \vec{B}\cdot\hat{n}da = \int_{\ell(t)} \vec{B}\cdot(d\vec{\ell} \times \vec{V}\Delta t) = \int_{\ell(t)} (\vec{V} \times \vec{B})\cdot d\vec{\ell}\Delta t$$

$$= \int_{S_1} [\nabla \times (\vec{V} \times \vec{B})]\cdot\hat{n}da\Delta t \qquad \text{(II.42)}$$

The last equality is an application of Stokes' theorem. Thus (II.42) becomes

$$\frac{dF}{dt} = \int_{S_1} \left[\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{V} \times \vec{B}) \right] \cdot \hat{n} da \qquad \text{(II.43)}$$

The frozen-in flux condition, eq. (II.38), then follows immediately from the hydromagnetic equation in the form of eq. (II.36).

The flux-preserving character of a plasma in the hydromagnetic limit imposes important constraints on possible fluid motions, which we will now enumerate. First define a magnetic flux tube to be a surface generated by moving any closed loop parallel to the magnetic field lines it intersects at a given instant. One creates this way a generalized cylindrical surface which encloses a constant amount of magnetic flux. By the definition it is clear that any patch of the surface of a flux tube encloses zero magnetic flux. Thus as a consequence of flux preservation, the fluid elements that form a flux tube at any instant, form a flux tube at all instants.

Flux preserving plasmas are also line preserving in the following sense. Imagine two fluid elements labeled A and B to be linked at time t by a magnetic field line. Now a field line can always be defined as the intersection of two flux tubes, and let us so define the field line linking A and B at time t. At this time the two fluid elements both belong to the surfaces of the two defining flux tubes. According to our previous corollary, they therefore must share the surfaces of two flux tubes in common at all times. That is, A and B must always lie at the intersection of two flux tubes, from which we may conclude that if two fluid elements are linked by a field line at any instant, they are always so linked. (For a fuller study see Stern 1966).

From the first corollary it is also easy to see that if a fluid element lies inside of a flux tube at one time, it always lies inside of it; and conversely if a fluid element lies outside of a flux tube at one time, it always lies outside of it.

II.6 Thawing of Magnetic Flux

If the generalized Ohm's law (II.6) in its abbreviated form (II.7) is solved in terms of $\vec{V} \times \vec{B}$ and the result substituted into the general expression for dF/dt (eq. II.43), we find with the aid of Faraday's induction law (I.35)

$$\frac{dF}{dt} = - \int_{S_1} (\nabla \times \vec{E}^*) \cdot \hat{n} da = - \oint_{\ell(t)} \vec{E}^* \cdot d\vec{\ell} = -EMF^* \qquad \text{(II.44)}$$

where EMF^* is the electromotive force per unit charge around $\ell(t)$ that the intrinsic electric field $\vec{E}^*$ produces. This result is not very surprising since it is formally the standard relation between time rate of change of magnetic flux and EMF. However, it is much more useful than a merely formal relation, because $\vec{E}^*$ is given explicitly in terms of the macroscopic plasma parameters by (II.8). Recall that the only approximations that entered into the derivation of (II.8) were $m_e \ll m_i$ and $\rho_c V \ll J$, both of which are well obeyed. Hence, if the freezing of the magnetic flux to the flow and its logical consequences are to be violated at any time or any place, the term or terms responsible for the thawing are contained in (II.8).

In the numerical example which was meant to typify solar system plasmas, the largest contributor to $\vec{E}^*$ was the Hall effect term. However as we saw in eq. (II.18) this term only moves the condition of freezing of the flux from the plasma as a whole to the electron gas component. That is, if we define F_e for the electron gas in the manner analogous to the definition of F for the plasma as a whole, namely the magnetic flux through a closed perimeter moving with the electron gas, then we can write in an obvious notation

$$\frac{dF_e}{dt} = - \int_{S_1} (\nabla \times E_e^*) \cdot \hat{n} da = - \int_{\ell_e(t)} \vec{E}_e^* \cdot d\vec{\ell}_e = - (\mathrm{EMF})_e^* \qquad \text{(II.45)}$$

in which

$$\vec{E}_e^{\,*} = \eta \vec{J} - \frac{1}{ne} \nabla \cdot \overleftrightarrow{P}_e + \frac{m_e}{e^2 n} \left[\frac{\partial \vec{J}}{\partial t} + \nabla \cdot (\vec{J}\,\vec{V} + \vec{J}\,\vec{V}) \right] \qquad \text{(II.46)}$$

is the intrinsic electric field in the frame of reference moving with the electron gas component of the plasma. If thawing of the magnetic flux from the electron gas component is to occur at any time or any place, the responsible terms are contained in (II.46).

The ohmic term is ordinarily thought of in connection with magnetic thawing, especially in collisonal plasmas. However, in the highly collisionless plasmas of space, the other two terms need to be considered, including the off-diagonal parts of the electron pressure tensor. These matters have been pursued in studies of magnetic merging,which entails strong but spatially localized violation of freezing. (For a review of this subject see Vasyliunas, 1975).

In the case of an isotropic electron pressure and a polytropic electron gas the second term in (II.46) becomes a pure gradient term and contributes nothing to $(\mathrm{EMF})_e^*$.

The notion of freezing of a vector flux to the plasma flow can be usefully reformulated at this level of retention of terms in the Ohm's law if certain flow parameters are incorporated in the frozen quantity, as will be shown in the next section.

II.7 The Generalized Vorticity Theorem

One of the most important properties of ordinary fluids is their tendency to preserve vorticity in the same sense that MHD fluids tend to preserve magnetic flux. In this section we determine whether MHD fluids have a vorticity conserving property in addition to their flux conserving property. First the definition of vorticity, $\vec{\omega}$, is needed

$$\vec{\omega} \equiv \nabla \times \vec{V} \qquad \text{(II.47)}$$

The vorticity theorem in ordinary hydrodynamics ($\vec{E} = \vec{B} = 0$) is derived for a non-viscous, polytropic fluid. Then the Euler equation (I.87) is simply

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\, \vec{V} + \nabla h^{*} = - \nabla \phi \qquad \text{(II.48)}$$

where ϕ is again the gravitational potential and the quantity h^{*} is the specific enthalpy defined by (I.88), which we repeat here

$$\nabla h^{*} \equiv \frac{\nabla p}{\rho} \qquad \text{(II.49)}$$

More generally, one may say that a function, h^{*}, that satisfies (II.49) exists if and only if p is a function of ρ. Such a fluid is called barotropic. In a barotropic fluid, surfaces of constant pressure coincide with surfaces of constant density. In general constant pressure surfaces and constant density surfaces need not coincide, and in such a case the fluid is termed baroclinic. In a baroclinic fluid, a function, h^{*}, satisfying (II.49) does not exist, and the preservation of vorticity, which as we shall see depends on its existance, does not obtain.

Take the curl of (II.48) after substituting from the vector identity (I.90) to find

$$\frac{\partial \vec{\omega}}{\partial t} = \nabla \times (\vec{V} \times \vec{\omega}) \qquad \text{(II.50)}$$

Equation (II.50) is formally identical to (II.36) with $\vec{B}$ replaced by $\vec{\omega}$. All of the consequences derived from it pertaining to $\vec{B}$ then apply without change to $\vec{\omega}$. In particular, in a non-MHD, barotropic, nonviscous fluid, the flux of vorticity (called the circulation, Γ) is preserved.

$$\Gamma \equiv \int \vec{\omega} \cdot \hat{n}\, da \qquad \text{(II.51)}$$

$$\frac{d\Gamma}{dt} = 0 \qquad \text{(II.52)}$$

in strict analogy with equations (II.37) and (II.38).

Let us now try to extend the result to the MHD situation by retaining the $\vec{J} \times \vec{B}$ force in the momentum equation. At first appearance, the attempt would seem certain to fail, because the derivation of (II.50) required all non-inertial terms in the momentum equation to vanish under the curl operator. There is no reason for $\nabla \times (\vec{J} \times \vec{B})$ to be zero. However, a generalized vorticity theorem can be obtained if we utilize a more exact form of the generalized Ohm's law. We retain the three largest terms according to the dimensional scale analysis given earlier.

$$\vec{E} = -\vec{V} \times \vec{B} - \frac{1}{ne} \nabla p_e + \frac{1}{ne} \vec{J} \times \vec{B} \qquad \text{(II.53)}$$

in which only the case of a scalar pressure is considered. Now by breaking the pressure explicitly into its two components, $p = p_i + p_e$, the Euler equation (I.87) can be written as

$$\frac{m_i}{e} \left[\frac{d\vec{V}}{dt} + \nabla h_i^* + \nabla \phi \right] = \frac{1}{ne} \left[-\nabla p_e + \vec{J} \times \vec{B} \right] \qquad \text{(II.54)}$$

in which we have used $\rho = m_i n$, with n the common electron and ion number density, and

$$\nabla h_i^* \equiv \frac{\nabla p_i}{\rho} \qquad \text{(II.55)}$$

Substitution of (II.54) into (II.53) gives

$$\vec{E} = -\vec{V} \times \vec{B} + \frac{m_i}{e} \left[\frac{d\vec{V}}{dt} + \nabla(\phi + h_i^*) \right] \qquad \text{(II.56)}$$

and thus

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{V} \times \vec{B}) - \frac{m_i}{e} \left[\frac{\partial \vec{\omega}}{\partial t} - \nabla \times (V \times \vec{\omega}) \right] \qquad \text{(II.57)}$$

Recombining then yields

$$\frac{\partial \vec{\Omega}}{\partial t} = \nabla \times (\vec{V} \times \vec{\Omega}) \qquad \text{(II.58)}$$

where

$$\Omega \equiv \vec{\omega} + \frac{e\vec{B}}{m_i} \qquad \text{(II.59)}$$

Note that $\dfrac{e\vec{B}}{m_i} = \vec{\omega}_i$ (II.60)

is the gyrofrequency of the ions. Hence

$$\vec{\Omega} = \vec{\omega} + \vec{\omega}_i \tag{II.61}$$

In most applications $\omega_i >> \omega$, that is, the vorticity of the fluid motion is much less than the ion gyrofrequency. In this limit we recover the flux-freezing law given by equation (II.36). If the opposite limit should ever occur, equation (II.50) reduces to the ordinary vorticity theorem.

In conclusion, one has either freezing of the magnetic field or of vorticity or of their properly weighted sum, but not of both separately.

II.8 The MHD Helmholtz Equation:

We have seen that in the MHD limit the amount of magnetic field inside of a closed loop of fluid elements is a constant of the motion. Also it is evident that the amount of mass inside of a closed volume of fluid elements is a constant of the motion. If there is no compression or stretching of the fluid in the direction parallel to $\vec{B}$, it must follow that the ratio of field density (i.e. B) and mass density is a constant. This kinetmatic relationship between the magnetic field and mass density is expressed formally and more generally in the MHD Helmholz equation, the derivation of which proceeds as follows. Write the continuity equation in the form (I.22)

$$\nabla \cdot \vec{V} = \rho \frac{d}{dt} \left(\frac{1}{\rho}\right) \tag{II.62}$$

Expand the curl of the vector cross product in equation (II.36) to arrive at (note $d\vec{B}/dt = \partial\vec{B}/\partial t + (\vec{V}\cdot\nabla)\,\vec{B}$)

$$\frac{d\vec{B}}{dt} + \vec{B}\,(\nabla\cdot\vec{V}) - (\vec{B}\cdot\nabla)\,\vec{V} = 0 \tag{II.63}$$

Divide (II.63) through by ρ, substitute in for $\nabla\cdot\vec{V}$ from (II.62) and recombine to find

$$\frac{d}{dt}\left(\frac{\vec{B}}{\rho}\right) - \left(\frac{\vec{B}}{\rho}\cdot\nabla\right)\vec{V} = 0 \tag{II.64}$$

This result with $\vec{B}$ replaced by the vorticity, $\vec{\omega}$, is known in hydrodynamics as the Helmholtz equation. Its formal similarity to the continuity equation is evident. To confirm the observation made at the beginning of this section, note that if $\vec{V}$ does not change in the direction of $\vec{B}$, then $\vec{B}/\rho$ is a constant of the motion.

Equation (II.64) has an important application to steady flows in which a stagnation point occurs (i.e. a point where $\vec{V} = 0$), for example the stagnation point in the solar wind at the magnetopause of planetary magnetospheres. In steady state, the equation may be rewritten as

$$(\vec{V}\cdot\nabla)\,\frac{\vec{B}}{\rho} = (\frac{\vec{B}}{\rho}\cdot\nabla)\,\vec{V} \qquad \text{(II.65)}$$

At an ordinary stagnation point, the right hand side is not zero because $\vec{V}$ is changing from one direction to the opposite direction across the stagnation point in the direction of $\vec{B}$. But the left hand side ought to be zero since $\vec{V} = 0$ there. To maintain the equality expressed by (II.65) it is necessary for ρ to vanish at the stagnation point. This is a characteristic of MHD flows which is not shared by ordinary fluids.

An alternative resolution of the dilemma has been suggested (see Sonnerup, 1980). Instead of forming a stagnation point, MHD flows may form stagnation lines such that $\vec{V} = 0$ along a finite stretch in the direction parallel to B. The stagnation line would terminate at both ends at a neutral point in the magnetic field (i.e. a point where $\vec{B} = 0$). Equation (II.65) is satisfied at a point which is both a stagnation point and a neutral point. Note that in parallel flows ($\vec{V}||\vec{B}$), this condition is met automatically and an ordinary neutral point can occur in the flow.

II.9 The Double Adiabatic Invariants

It is possible in the hydromagnetic limit to derive prognostic equations for the scalars $p_\parallel$ and $p_\perp$ of the anisotropic pressure tensor (I.27) in the manner in which equation (I.57) was derived for the scalar pressure p. The strictly analogous procedure entails algebra too lengthy for inclusion in this chapter. However, a kinetic theory argument will be given here that results in the correct forms of adiabatic invariants for $p_\perp$ and $p_\parallel$ which correspond to the adiabatic form $P/\rho^{5/3}$ for the scalar pressure.

Consider a particle enclosed in a container, one wall of which is a piston, which can be moved in and out in order to change the volume of the container at will. The container will be used to simulate adiabatic changes in the three equal components of the isotropic pressure tensor of a collision dominated gas and the two perpendicular components and the one parallel component of the anisotropic tensor of a collisionless magnetized plasma. In the first instance the energy gained (or lost) by the particle in colliding with the moving wall will be shared by the three components equally to preserve isotropy. In the second instance the energy change will be shared equally by the two components of $p_\perp$. In the third instance the one component of $p_\parallel$ will retain the entire change.

The adiabatic condition is imposed by moving the wall slowly compared to the speed of the particle in the container, and the collision between the particle and all of the walls is assumed to be perfectly elastic.

Let the velocity $\vec{v}$ of the particle be decomposed into cartesian (x,y,z) components with the x-axis parallel to the motion of the piston. Let u be the speed of the piston measured positive when the piston moves in the direction to decrease the volume of the container (compression). Then the change in v_x after one collision with the moving wall is

$$v_x(\text{after}) = v_x(\text{before}) + 2u \tag{II.66}$$

The corresponding <u>first order</u> energy change Δw_x is then

$$\Delta w_x = \frac{1}{2} m \left[v_x^2(\text{after}) - v_y^2(\text{before})\right] = 2muv_x \tag{II.67}$$

Let ν be the number of components that share the energy acquired in one collision before a second collision with the moving wall occurs. Thus $\nu = 3$ for the isotropic gas, $\nu = 2$ for $p_\perp$ and $\nu = 1$ for $p_{\parallel}$. After the sharing of the energy takes place, the net change in Δw_x is

$$\Delta w_x = \frac{2}{\nu} muv_x \tag{II.68}$$

Now the number of collisions the particle has with the moving wall each second f_{coll} is given by

$$f_{coll} = \frac{v_x}{2L_x} \tag{II.69}$$

where L_x is the separation between the face of the piston and the stationary wall opposite from it. Thus

$$\frac{dw_x}{dt} = \frac{1}{\nu}\frac{u}{L_x} mv_x^2 = \frac{2}{\nu}\frac{u}{L_x} w_x \tag{II.70}$$

But by the definitions of u and L

$$u = -\frac{dL_x}{dt} \tag{II.71}$$

Substitution of (II.71) into (II.70) and subsequent integration give

$$\frac{d}{dt} \ell n \left(w_x L_x^{2/\nu}\right) = 0 \tag{II.72}$$

The length L_x is related to the mass density ρ by the expression

$$\rho = \frac{mN}{AL_x} \tag{II.73}$$

in which the total number of particles in the container, N, the area of the container in the yz-plane, $A = L_y L_z$, and the mass of the particles, m, remain constant during the motion of the piston.

The energy in the x-component of motion is related to the temperature of the gas and consequently to the pressure by

$$w_x = \frac{1}{2} k T_x = \frac{1}{2} \frac{m p_x}{\rho} \qquad \text{(II.74)}$$

where (I.62) and (I.63) have been used.

Consider first the isotropic case. Then combining (II.71) with $\nu = 3$ and (II.72) and (II.74) gives

$$\frac{d}{dt} \ln \left(\frac{p}{\rho^{5/3}}\right) = 0 \qquad \text{(II.75)}$$

Thus we recover the correct result for adiabatic changes in an ideal monatomic isotropic gas (cf. eq. I.57).

To treat the case of the magnetized collisionless plasma, it is necessary to recognize that two of the walls of our container form a magnetic flux tube. Thus for $p_\perp$, we must put $\nu = 2$ in (II.72) but also the equation for L_x becomes

$$F = B L_x L_y \qquad \text{(II.76)}$$

where F is the magnetic flux enclosed by the container, which must remain constant as the piston moves, by the freezing laws described in Section II.5. The distance L_y is also constant by design. Hence, after substituting into (II.72) from (II.76) and dropping derivatives of quantities that remain constant as the piston moves, there results

$$\frac{d}{dt} \ln \left(\frac{w_x}{B}\right) = 0 \qquad \text{(II.77)}$$

This is seen to be the expression for the constancy of the first adiabatic invariant (I.111), derived here by a statistical mechanics argument. Equation (II.74) relating W_x, pressure and mass density can now be used to convert (II.77) into the expression for the first of the two adiabatic invariants of magnetohydrodynamics

$$\frac{d}{dt} \ln \left(\frac{p_\perp}{\rho B}\right) = 0 \qquad \text{(II.78)}$$

The expression for $p_\parallel$ is found by setting $\nu = 1$ in (II.72). Also since the motion of the piston in this case is parallel to the flux tube, the equation for L_x is again (II.73), but with $A = F/B$, where F is constant. The energy is expressed in terms of pressure by (II.74). Making the indicated substitutions and dropping derivatives of constant quantities result in

$$\frac{d}{dt} \ln \left(\frac{p_{\parallel} B^2}{\rho^3}\right) = 0 \qquad \text{(II.79)}$$

The quantities

$$\alpha_{\perp} \equiv \frac{p_{\perp}}{\rho B} \qquad \text{(II.80)}$$

$$\alpha_{\parallel} \equiv \frac{p_{\parallel} B^2}{\rho^3} \qquad \text{(II.81)}$$

are the two adiabatic invariants of collisionless MHD. In adiabatic MHD flows, $\alpha_{\perp}$ and $\alpha_{\parallel}$ are constants of the motion. The discussion relating to the quantity α defined in (I.82) applies to $\alpha_{\perp}$ and $\alpha_{\parallel}$ as well.

Note that B can be eliminated from (II.80) and (II.81) by combining them into a hybrid adiabatic invariant

$$(\alpha_{\perp}^2 \alpha_{\parallel})^{1/3} = \frac{(p_{\perp}^2 p_{\parallel})^{1/3}}{\rho^{5/3}} \qquad \text{(II.82)}$$

In the case $p_{\perp} = p_{\parallel}$, (II.82) reverts to the expression for the adiabatic invariant for an isotropic pressure (II.75).

A more important hybrid combination of the (II.80) and (II.81) is the expression for the pressure ratio

$$\frac{p_{\perp}}{p_{\parallel}} = \frac{\alpha_{\perp}}{\alpha_{\parallel}} \frac{B^3}{\rho^2} \qquad \text{(II.83)}$$

The ratio B^3/ρ^2 is not constant for any known flow in solar system MHD and is perhaps never constant in natural plasmas. It is therefore a fundamental property of collisionless MHD flows to become anisotropic. The sense of the anisotropy ($p_{\perp} > p_{\parallel}$ or $p_{\parallel} > p_{\perp}$) depends on how the ratio B^3/ρ^2 changes as the flow progresses. It will be seen in Section IV that a plasma becomes unstable if either sense of anisotropy gets too large. Thus, there is a tendency for collisionless MHD flows to evolve toward instabilities driven by the pressure anisotropy.