III. MHD WAVES AND DISCONTINUITIES

III.1 Linearized Plane Waves in an Isotropic Magnetized Plasma

A significant portion of the phenomenology of MHD plasmas concerns the presence of waves and discontinuities. We look first at the kinds of small amplitude isentropic waves that can propagate in a homogeneous, uniform magnetized plasma with isotropic pressure. Gravity will be ignored, which excludes the possibility of coupled gravity-MHD waves.

The relevant equations are

Continuity eq. $$\frac{\partial \rho}{\partial t} + (\vec{V}\cdot\nabla)\,\rho + \rho\nabla\cdot\vec{V} = 0 \qquad \text{(III.1)}$$

Euler eq. $$\rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}\right] + \nabla p = \vec{J} \times \vec{B} \qquad \text{(III.2)}$$

Isentropic eq. $$p = \alpha\rho^{\gamma} \qquad \text{(III.3)}$$

Maxwell's eqs. $$\nabla \times \vec{B} = \mu_o \vec{J} \qquad \text{(III.4)}$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla\times(\vec{V} \times \vec{B}) \qquad \text{(III.5)}$$

The electric field has been dropped where appropriate by the approximations given in Section II.2 and eliminated from Faraday's induction law by use of the hydromagnetic approximation. The pressure p and current density $\vec{J}$ can be eliminated by direct substitution into the Euler equation.

Euler eq. $$\rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}\right] + \alpha\nabla\rho^{\gamma} = -\frac{1}{\mu_o}\vec{B}\times(\nabla\times\vec{B}) \qquad \text{(III.6)}$$

Equations (III.1, 5 and 6) are a complete, deterministic set for the variables ρ, $\vec{V}$ and $\vec{B}$, from which p and $\vec{J}$ can be considered derived quantities.

These equations will now be linearized and subjected to an usual plane wave expansion. Denote zero-order quantities by subscripted zeros and first-order quantities by a prefixed δ. Then

$$\rho = \rho_o + \delta\rho \qquad \text{(III.7)}$$

$$\vec{V} = \vec{V}_o + \delta\vec{V} \qquad \text{(III.8)}$$

$$\vec{B} = B_o + \delta\vec{B} \qquad \text{(III.9)}$$

Zero-order quantities are constant in space and time. Quadratic and higher order terms in first-order quantities will be dropped. The condition that first-order quantities propagate as plane waves is imposed by the relation

$$\delta Q \rightarrow \delta Q \; e^{i(\omega' t - \vec{k}\cdot\vec{r})} \tag{III.10}$$

where δQ denotes an arbitrary first-order quantity, and its value on the right hand side is now regarded as a constant amplitude to the plane wave. The prime on ω denotes the Doppler frequency. Since we have allowed the possibility of a zero-order velocity of the medium, the frequency of the waves will be Doppler shifted in our frame of reference.

It will be useful to carry out the calculation in a cartesian coordinate system with the z-axis defined to be parallel to $\vec{B}_o$ and the xz-plane defined to contain the propagation vector $\vec{k}$ (Fig. III.1). No loss of generality is incurred by this choice.

$$\vec{B}_o = B_o \hat{z} \tag{III.11}$$

$$\vec{k} = k_x \hat{x} + k_z \hat{z} \tag{III.12}$$

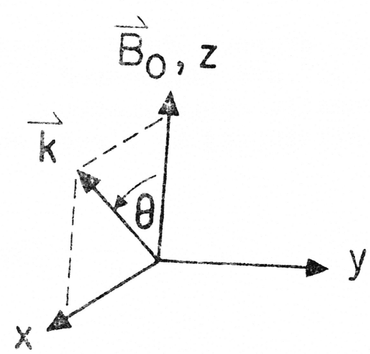

Figure III.1 Coordinate system for describing MHD plane waves

Substitution of (III.8 through 10) into (III 1, 5 and 6) and making the plane wave replacements, $\partial/\partial t \rightarrow i\omega'$, $\nabla Q = -i\vec{k}Q$, $\nabla\cdot\vec{Q} = -ik\cdot\vec{Q}$ and $\nabla \times \vec{Q} = -i\vec{k} \times Q$, result in

$$(\omega' - \vec{k}\cdot\vec{V}_o)\frac{\delta\rho}{\rho_o} = \vec{k}\cdot\delta\vec{V} \tag{III.13}$$

$$(\omega' - \vec{k}\cdot\vec{V}_o)\,\delta\vec{V} = C_S^2 \frac{\delta\rho}{\rho_o}\vec{k} + C_A^2\, \hat{z} \times (\vec{k} \times \frac{\delta\vec{B}}{B_o}) \tag{III.14}$$

$$(\omega' - \vec{k}\cdot\vec{V}_o)\ \frac{\delta\vec{B}}{B_o} = (\vec{k} \times \hat{y})\ \delta V_x - k_z \delta V_y\ \hat{y} \qquad \text{(III.15)}$$

Where the parameters C_S and C_A, are characteristic velocities of the medium and are defined by

$$C_S^2 \equiv \gamma\alpha\rho^{\gamma-1} = \gamma\ \frac{p}{\rho} \qquad \text{(III.16)}$$

$$C_A^2 \equiv \frac{B_o^2}{\mu_o \rho_o} \qquad \text{(III.17)}$$

Equation (III.16) is the usual definition of the speed of sound. The meaning of (III.17) is given later. To arrive at (III.15) it is necessary to expand the curl of the cross product with the vector identity

$$\nabla \times (\vec{V} \times \vec{B}) = (\vec{B}\cdot\nabla)\ \vec{V} - (\vec{V}\cdot\nabla)\ \vec{B} + \vec{V}\ (\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{V}) \qquad \text{(III.18)}$$

and use the condition $\nabla\cdot\vec{B} = 0$.

The coefficients on the left hand sides of (III.13-15) show that the Doppler frequency ω' is related to the frequency in the plasma rest frame ω by

$$\omega' = \omega + \vec{k}\cdot\vec{V}_o \qquad \text{(III.19)}$$

In the remainder of the section, we will use the plasma rest frame frequency ω.

The variables $\delta\rho$ and $\delta\vec{B}$ can be eliminated by substitution to produce a single equation for $\delta\vec{V}$

$$\omega^2 \delta\vec{V} = C_S^2(\vec{k}\cdot\delta\vec{V})\vec{k} + C_A^2\ (k^2 \delta V_x \hat{x} + k_z^2 \delta V_y\ \hat{y}) \qquad \text{(III.20)}$$

The three components of (III.20) can now be written to provide three equations for the three components of $\delta\vec{V}$. Since the equations are homogeneous, non-trivial solution exist only when the determinant of the coefficient matrix vanishes. This then provides the dispersion equation for the waves.

The resulting dispersion equation will be cubic in ω^2, implying the existence of three wave modes, unless some are degenerate. One mode can be isolated already at this point, leaving only two modes to be obtained by the formal approach. The y-component of (III.20) involves only δV_y, and is therefore a pure mode, the dispersion equation for which can be seen to be

$$\omega_i^2 = C_A^2 k_z^2 \qquad \text{(III.21)}$$

where the subscript i on ω in this instance designates the intermediate mode to distinguish this one from the two others that are described below. The phase velocity of the wave which in general is given by

$$\vec{V}_{ph} = \frac{\omega}{k} \hat{k} \qquad \text{(III.22)}$$

in the case of (III.21) becomes

$$(\vec{V}_{ph})_i = C_A \cos\theta \, \hat{k} \qquad \text{(III.23)}$$

where θ is the angle between $\vec{k}$ and $\vec{B}_o$ (see Figure III.1).

This very simple mode has a number of interesting properties. In terms of the velocity perturbation, it is a purely transverse mode, both to $\vec{k}$ and to $\vec{B}_o$, since δV_y is orthogonal to both of these vectors. From (III.15) it is seen that $(\delta\vec{B})_i = \delta B_y \hat{y}$ is also transverse to $\vec{k}$ and $\vec{B}_o$. The mode does not entail a perturbation in density, pressure or magnetic field strength. That is

$$(\delta\rho)_i = (\delta B)_i = 0 \qquad \text{(III.24)}$$

The fact that $(\delta\rho) = 0$ follows from (III.13) with $\vec{k}\cdot(\delta\vec{V})_i = 0$. It then follows from (III.3) that there is then no perturbation in pressure. The perturbation in field strength is given in general by

$$\frac{\delta B}{B_o} = \frac{\delta B_z}{B_o} \qquad \text{(III.25)}$$

as can be seen by expanding $(\vec{B}_o + \delta\vec{B})^2/B_o^2$ to first order and comparing the result with $(B_o + \delta B)^2/B_o^2$. Since $(\delta B_z)_i = 0$ for this mode, $(\delta B)_i = 0$ also.

The energy carried by the mode propagates strictly parallel to $\vec{B}_o$ at the characteristic speed C_A. The velocity of energy propagation is the group velocity, which is defined in general by

$$\vec{V}_g = \frac{d\omega}{d\vec{k}} = \frac{\partial\omega}{\partial k_x}\hat{x} + \frac{\partial\omega}{\partial k_y}\hat{y} + \frac{\partial\omega}{\partial k_z}\hat{z} \qquad \text{(III.26)}$$

From (III.21) there results

$$(\vec{V}_g)_i = \pm C_A \hat{z} \qquad \text{(III.27)}$$

The relationship between $(\delta\vec{B})_i$ and $(\delta\vec{V})_i$ is useful in data interpretation applications to determine the direction of propagation of the wave. From (III.15)

$$\omega_i \frac{(\delta B_y)_i}{B_o} = - k_z (\delta V_y)_i \qquad \text{(III.28)}$$

or

$$k_z = - \frac{\omega_i}{B_o} \frac{(\delta B_y)_i}{(\delta V_y)_i} \qquad \text{(III.29)}$$

Thus, if the magnetic field and velocity perturbations are in the same direction (i.e. both in the +y or both in the -y direction at a given point and time), the wave energy is propagating antiparallel to $\vec{B}_o$. If they are in opposite directions, the wave energy is propagating parallel to $\vec{B}_o$

The mode we have just described was discovered by Alfven who recognized that its basic properties result from a balance between the inertial response of the mass of the plasma and the magnetic tension resulting from stretching a field line. The characteristic speed C_A is called the <u>Alfvén speed</u>. As already noted, in the context of the three wave modes of MHD, this mode is called the intermediate wave and C_A is the intermediate wave speed.

The dispersion relation for the other two modes is found by writing out the x and z-components of equation (III.20), which gives after some rearranging

$$(\omega^2 - C_S^2 k_x^2 - C_A^2 k^2)\ \delta V_x - C_S^2 k_x k_z \delta V_z = 0 \qquad \text{(III.30)}$$

$$- C_S^2 k_x k_z \delta V_x + (\omega^2 - C_S^2 k_z^2)\ \delta V_z = 0 \qquad \text{(III.31)}$$

Setting the determinant of the coefficient matrix to zero and solving for $\omega^2/k^2 = V_{ph}^2$, we find

$$(V_{ph}^2)_{f,s} = \frac{1}{2} [(C_S^2 + C_A^2) \pm \sqrt{(C_S^2 + C_A^2)^2 - 4C_S^2\, C_A^2 \cos^2\theta}\,] \qquad \text{(III.32)}$$

The two modes are differentiated by their phase speeds and accordingly are referred to as the <u>fast and slow modes</u>. It is instructive to consider the propagation of these waves parallel to $\vec{B}_o (\theta = 0)$ and perpendicular to $\vec{B}_o$ $(\theta = \pi/2)$ as special cases.

$$\vec{k}\,||\,\vec{B}_o \;:\; (V_{ph}^{2})_{f,s} = \begin{cases} C_S^2 \\ C_A^2 \end{cases} \qquad \text{(III.33)}$$

$$\vec{k} \perp \vec{B}_o : (V_{ph}^{2})_{f,s} = \begin{cases} C_S^2 + C_A^2 \\ 0 \end{cases} \qquad \text{(III.34)}$$

A wave that propagates parallel to $\vec{B}_o$ is seen to be either a pure sound wave (which is a longitudinal wave $\delta\vec{V}||\vec{k}$) or a pure Alfven wave (which is a transverse wave $\delta\vec{V}\perp\vec{k}$). The phase speed of a fast mode wave for which $\vec{k}\perp\vec{B}_o$ is the Pythagorean sum of the sound speed and the Alfven speed. The slow mode does not propagate perpendicular to $\vec{B}_o$, which is also true of the intermediate mode, as can be seen from (III.23).

To demonstrate that the intermediate mode phase speed is indeed intermediate between the fast and slow phase speeds, define the two velocity ratios

$$R_{\pm} \equiv \frac{2C_A^2\cos^2\theta}{(C_S^2+C_A^2) \pm \sqrt{(C_S^2+C_A^2)^2 - 4C_A^2 C_S^2\cos^2\theta}} \equiv \frac{A}{S \pm \sqrt{S^2-D}} \qquad \text{(III.35)}$$

where the symbols A, S and D are introduced for brevity and are defined by their positions in context. We want to show that unity lies between R_+ and R_-, that is, one is greater than and one is less than unity. The possibility that either ratio may also be unity exists in the limits $\theta = 0$ and $\theta = \pi/2$. Unity lies between R_+ and R_- if and only if

$$(R_+ - 1)(R_- - 1) \leq 0 \qquad \text{(III.36)}$$

In terms of A, S and D (III.36) is

$$\left(\frac{A-S-\sqrt{S^2-D}}{S+\sqrt{S^2-D}}\right)\left(\frac{A-S+\sqrt{S^2-D}}{S-\sqrt{S^2-D}}\right) = \frac{A^2-2AS+D}{D} \leq 0 \qquad \text{(III.37)}$$

Substituting back the original variables, we find

$$4C_A^4\,(\cos^2\theta-1)\cos^2\theta \leq \theta \qquad \text{(III.38)}$$

That (III.38) is an obviously true statement verifies that the ordering of phase velocities implied by the names of the three MHD wave modes is correct.

Figure III.2 shows four examples of the relative phase speeds of the three MHD modes and how they depend on the angle between $\vec{k}$ and $\vec{B}_o$.

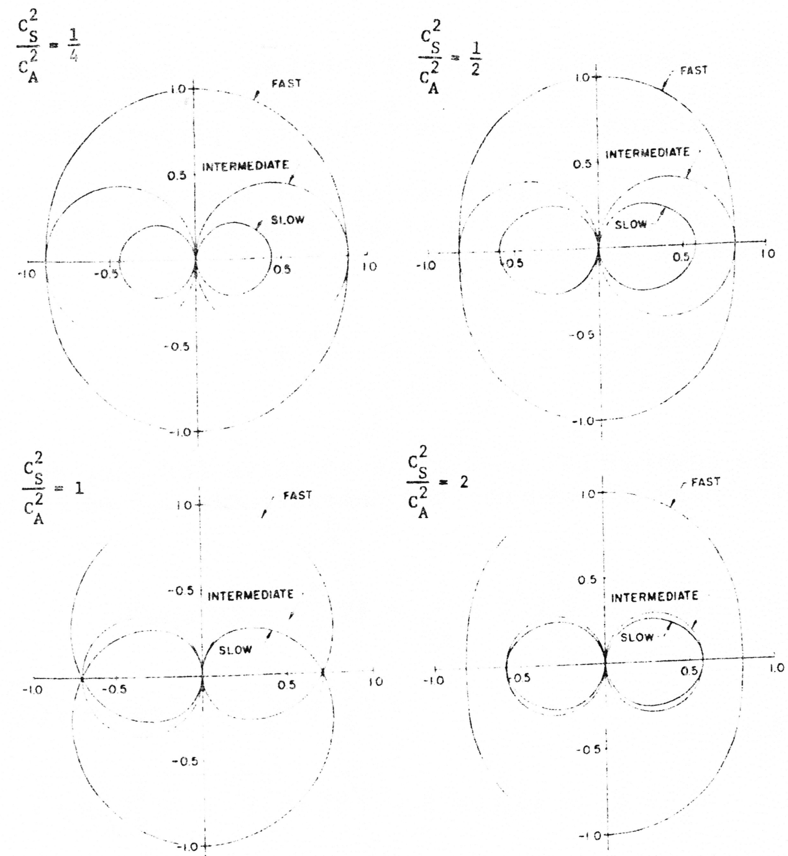

Figure III.2 Friedrichs Diagram. Polar plots showing the dependence of the propagation speeds of the three linear wave modes on the angle between the wave normal $\hat{k}$ and the magnetic field $\vec{B}_o$ for several values of the ratio of sound speed C_S to Alfvén speed C_A. In these plots B_θ is parallel to the horizontal axis. Speeds have been normalized with respect to $\sqrt{C_S^2 + C_A^2}$ (From Kantrowitz and Petschek, 1964).

In contrast to the intermediate mode, both the fast and slow modes are compressive. That is, they entail changes in density (and hence pressure) and changes in the field strength. Equations (III.13, 15 and 31) can be combined to give the dependence of $(\delta\rho)_{f,s}$ and $(\delta B)_{f,s}$ on the velocity perturbations.

$$C_S^2 \frac{(\delta\rho)_{f,s}}{\rho_o} = \frac{\omega_{f,S}}{k_z} (\delta V_z)_{f,s} \qquad \text{(III.39)}$$

$$\omega_{f,S} \frac{(\delta B)_{f,s}}{B_o} = k_x (\delta V_x)_{f,s} \qquad \text{(III.40)}$$

Mass compression (and rarefaction) is seen to result from velocity perturbations parallel to $\vec{B}_o$ and field compression (and rarefaction) to result from velocity perturbations perpendicular to $\vec{B}_o$

As a last characteristic of fast and slow modes, we find with the use of (III.31) a single expression relating the density and field strength perturbations.

$$\frac{(\delta\rho)_{f,s}}{\rho_o} = \frac{\omega_{f,s}^2}{\omega_{f,s}^2 - C_S^2 k_z^2} \frac{(\delta B)_{f,s}}{B_o} \qquad \text{(III.41)}$$

The combination $\omega^2 - C_S^2 k_z^2$ is positive for fast mode waves and negative for slow mode waves, the demonstration of which is as follows

$$(\omega_{f,s}^2 - C_S^2 k_z^2) \times \frac{2}{k^2} = 2(V_{ph}^2) - 2C_S^2 \cos^2\theta$$

$$= C_S^2 + C_A^2 - 2C_S^2\cos^2\theta \pm \sqrt{(C_S^2+C_A^2)^2 - 4C_S^2C_A^2\cos^2\theta}$$

But

$$C_S^2 + C_A^2 - 2C_S^2\cos^2\theta = \sqrt{(C_S^2+C_A^2)^2 - 4C_S^2C_A^2\cos^2\theta - 4C_S^4\cos^2\sin^2\theta}$$

$$\leq \sqrt{(C_S^2+C_A^2)^2 - 4C_S^2C_A^2\cos^2\theta} .$$

An important distinction between the fast and slow modes in thereby revealed. Density and field strength perturbations are in phase for the fast mode but out of phase for the slow mode. Correlations between density and field strength oscillations can be used as a diagnostic to identify the mode type.

Figure (III.3) indicates the geometrical basis for the variation in field strength. The top figure depicts the field configuration of an intermediate mode. The propagation vector makes an arbitrary angle in the xz-plane, perpendicular to the paper, such that the wave fronts intersect the plane of the paper in lines parallel to the y-axis. It is evident from the equidistant spacing of the field lines that this mode is non-compressive. The bottom figure shows a wave propagating in

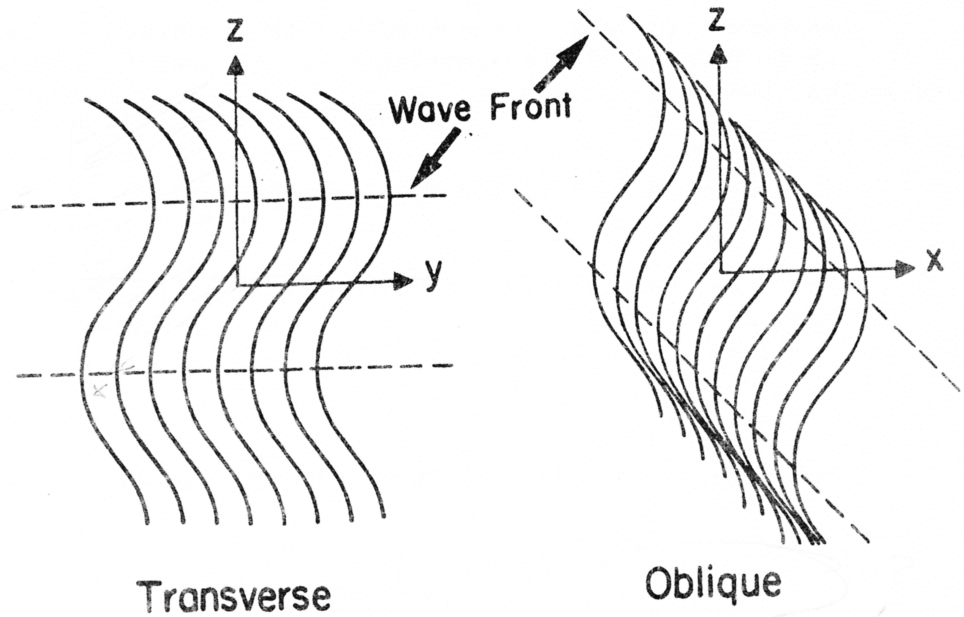

Figure III.3 The compression and rarefaction of magnetic field strength is determined geometrically by the alignment of wave crests and troughs along horizontal or oblique wave fronts.

the same plane as the field perturbation, such as is true for both the fast and slow modes. The fact that the wave crests lie along the oblique lines that represent the wave fronts geometrically causes the field lines to lie close together in one part of the wave and to be relatively separated in another part. In this picture the fast and slow modes are distinguished by whether the density compressions coincide with the strips where the field lines are close together or with the strips where they are farther apart.

An important type of problem arises in solar system plasmas when magnetic field lines connect plasmas that are in motion relative to each other. Examples of such situations are found in the motion of the solar wind relative to the sun and the motions of the plasmas of a planetary magnetosphere relative to the planetary ionosphere. The magnetic field plays the role of an elastic medium in MHD and a ponderomotive force is generated which acts to reduce the relative motion. In order that the ponderomotive force act equally and oppositely in the two plasmas, the associated electrical current must link both bodies. The current is therefore required to flow back and forth parallel to the magnetic field that connects the two regions. The currents are carried by MHD waves, which in a steady state situation will be standing waves but otherwise they will be propagating waves attached to one or the other or both of the coupled plasmas. It is readily shown that the only MHD wave capable of carrying an electrical current parallel to the magnetic field is the obliquely propagating intermediate mode waves. Thus, the oblique intermediate wave is responsible for the mechanical coupling of magnetically linked plasmas. The verification of this statement follows from MHD wave form of Faraday's induction law (III.15) which because of its importance to the present demonstration is rewritten here

$$\omega \frac{\delta \vec{B}}{B_o} = (\vec{k} \times \hat{y})\ (\delta V_x)_{f,s} - k_z (\delta V_y)_i \hat{y} \qquad \text{(III.42)}$$

Now from the perturbation form of (II.4) the current carried by the wave is

$$\delta \vec{J} = -\frac{i}{\mu_o} \vec{k} \times \delta \vec{B} \qquad \text{(III.43)}$$

Substitution of (III.42) into (III.43) yields

$$\delta \vec{J} = \frac{iB_o}{\mu_o \omega} k^2 (\delta V_x)_{f,s}\ \hat{y} + \frac{iB_o}{\mu_o \omega} k_z (\delta V_y)_i\ (-k_z \hat{x} + k_x \hat{z}) \qquad \text{(III.44)}$$

This first term of the right hand side is the current associated with fast and slow waves, and the second term is the current associated with the intermediate wave. One sees that only the second term has a vector component parallel to $\vec{B}_o$, i.e. parallel to $\hat{z}$. The amplitude of the parallel component is

$$(\delta \vec{J})_{\parallel} \equiv \hat{z} \cdot \delta \vec{J} = \frac{iB_o}{\mu_o \omega} k_x k_z (\delta V_y)_i = \frac{iB_o}{\mu_o \omega} k^2 (\delta V_y)_i \sin\theta \cos\theta \qquad \text{(III.45)}$$

or in terms of the perturbation magnetic field (III.28)

$$(\delta \vec{J})_{\parallel} = -\frac{i}{\mu_o} k (\delta B_y)_i \sin\theta \qquad \text{(III.46)}$$

The factor $\underline{i}$ in the coefficient means that in the context of sinusoidally oscillating plane waves, $(\delta \vec{J})_{\parallel}$ is ninety degrees out of phase with $(\delta V_y)_i$ and $(\delta B_y)_i$. Note that $(\delta \vec{J})_{\parallel} = 0$ when $\theta = 0$ that is in the case of parallel propagation. Thus, only obliquely propagating waves carry parallel current.

The properti4s of the MHD waves become significantly modified when the pressure is anisotropic. The dispersion relation for an anisotropic plasma is given in Section IV.1 on the firehose and mirror instabilities.

III.2 MHD Discontinuities

In the previous section the differential MHD equations were transformed into a set of algebraic relations by first linearizing them and then making the assumption that they had plane wave solutions. The procedure lead to a description of the three MHD wave modes. A different set of algebraic relations can be found to describe the opposite limit, that is discontinuous rather than small amplitude variations.

Assume that a sudden change in the MHD parameters occurs across a surface in space, the local radius of curvature R_D of which is much greater than the local thickness δ_D. In effect, this is a statement of what is meant by a discontinuity. Then there exists a scale regime intermediate between δ_D and R_D in which the MHD parameters may be considered to be uniform on either side of the surface but to change discontinuously across it.

If the discontinuity is moving with (non-relativistic) velocity $\vec{V}_S$ relative to a frame of reference in which the MHD parameters are specified and in particular in which the flow velocity is $\vec{V}$, it is always possible to make a Galilean transformation into a frame of reference in which the discontinuity is at rest. In the rest frame of the discontinuity, the flow velocity, $\vec{U}$, is

$$\vec{U} = \vec{V} - \vec{V}_S \qquad \text{(III.47)}$$

The conservation form of the MHD equations are particularly well suited for the investigation of changes across a discontinuity. In the rest frame of the discontinuity, we may assume time independence. Then the conservation equations for mass, momentum and energy (I.21, 46 and 48) become

$$\nabla\cdot(\rho\vec{U}) = 0 \qquad \text{(III.48)}$$

$$\nabla\cdot(\rho\vec{U}\vec{U} + \overleftrightarrow{P} - \overleftrightarrow{T}) = 0 \qquad \text{(III.49)}$$

$$\nabla\cdot[(\tfrac{1}{2}\rho U^2 + u + \overleftrightarrow{P})\cdot\vec{U} + \vec{S}] = 0 \qquad \text{(III.50)}$$

The gravitational terms have been dropped, as in the previous study of MHD waves. The heat flux vector has also been dropped in the energy equation, although in the case of compressive, collisionless MHD shocks, it could represent an important sink of energy. The electric field terms will be dropped from the Maxwell stress tensor, in accordance with the approximations discussed in Section II.2. The anisotropic form of the pressure tensor (eq. I.27) will be retained, but examples will be given also for the case of a scalar pressure. The internal energy corresponding to the anisotropic pressure is

$$u = p_\perp + \frac{1}{2}p_\parallel \qquad \text{(III.51)}$$

The differential equations (III.48-50) are converted to algebraic equations by integrating them over a volume in the shape of a thin cylinder with unit area faces aligned parallel to the surface of the discontinuity and bracketing it, such that the discontinuity cuts completely through the side of the cylinder, dividing it into equal parts. The cylinder is designed such that the area of the side is negligible compared to unity. Gauss' law is then used to replace the volume integrals by an integral over the surface of the cylinder. The contribution from the side is negligible by design, and since the unit area faces lie in the regions of constant parameters on the two sides of the discontinuities, the integrals over them are the integrals themselves. The final result is then

$$\hat{n}_1 \cdot \vec{Q}_1 + \hat{n}_2 \cdot \vec{Q}_2 = 0 \qquad \text{(III.52)}$$

in which subscripts 1 and 2 distinguish between quantities evaluated on the two sides of the discontinuity, $\hat{n}$ is the outward pointing unit normal to the cylinder faces, and $\vec{Q}$ designates any of the three composite quantities on which the divergence operator operates in eqs. (III.48-50). In the case of (III.49), Q is a tensor.

Since by construction $\hat{n}_1 = -\hat{n}_2$, let

$$\hat{n} \equiv \hat{n}_2 = -\hat{n}_1 \qquad \text{(III.53)}$$

replace $\hat{n}_1$ and $\hat{n}_2$ in (III.52), and define the difference operator $[[Q]]$ by

$$[[Q]] \equiv Q_1 - Q_2 \qquad \text{(III.54)}$$

Then the equation for the conservation of mass, momentum and energy take the form

$$[[\rho U_n]] = 0 \qquad \text{(III.55)}$$

$$[[\rho U_n \vec{U} + (\overleftrightarrow{P} - \overleftrightarrow{T}) \cdot n]] = 0 \qquad \text{(III.56)}$$

$$[[(\frac{1}{2} \rho U^2 + 2p_\perp + \frac{1}{2} p_\parallel + (p_\parallel - p_\perp) \frac{B_n^2}{B^2}) U_n + S_n]] = 0 \qquad \text{(III.57)}$$

where a subscripted n denotes components parallel to $\hat{n}$. From the magnetohydrodynamic expression for the Poynting vector (II.28) we can write S_n in the form

$$S_n = \frac{B_t}{\mu_o}(B_t U_n - B_n U_t) \tag{III.58}$$

To this set of equations we must add the difference equations that result from integrating the two Maxwell equations $\nabla\cdot\vec{B} = 0$ and $\nabla x\vec{E} = 0$ (in steady state) across the discontinuity in a manner analogous to the treatment of the conservation equations. In the case of $\nabla x\vec{E} = 0$ the domain of integration is the area of a thin strip with unit length sides parallel to and bracketing the discontinuity, and Stoke's theorem is used. The result is

$$[[B_n]] = 0 \tag{III.59}$$

$$[[E_t]] = 0 \tag{III.60}$$

where $E_t = \vec{E}\cdot\hat{t}$ and $\hat{t}$ is a arbitrary unit vector tangent to the surface of the discontinuity ($\hat{t}\cdot\hat{n} = 0$). Replacing the electric field by $-\vec{V}x\vec{B}$ gives in place of (III.60)

$$[[U_n B_t - U_t B_n]] = 0 \tag{III.61}$$

It is useful to decompose the vector equation (III.56) into components parallel and perpendicular to $\hat{n}$. Scalar multiplication of (III.56) with $\hat{n}$ and $\hat{t}$ results in

$$[[\rho U_n^2 + (p_{\parallel} - p_{\perp})\frac{B_n^2}{B^2} + p_{\perp} + \frac{B_t^2}{2\mu_o}]] = 0 \tag{III.62}$$

$$[[\rho U_n U_t - \xi\frac{B_n B_t}{\mu_o}]] = 0 \tag{III.63}$$

in which the combination

$$\xi \equiv 1 - \frac{p_{\parallel} - p_{\perp}}{B^2/\mu_o} \tag{III.64}$$

appears so frequently, it is given its own symbol, and we have used the relation

$$[[B^2]] = [[B_n^2 + B_t^2]] = [[B_t^2]] \tag{III.65}$$

that follows from (III.59).

The six equations (III.55, 57, 59, 61, 62 and 63) form the complete set of continuity relations available from the conservation equations and Maxwell's equations. If the problem of interest is to determine the parameters on one side of the discontinuity given the parameters on the other side, there are seven unknowns ρ, U_n, U_t, $p_\perp$, $p_\parallel$, B_t, and B_t. If the pressure is isotropic so that it contributes one scalar to the list of unknowns instead of two, the set is completely deterministic. Otherwise an additional relation must be imposed by theory or measurement to close the set.

There exists a classification scheme for MHD discontinuities that divides them according to whether or not the plasma flows through the discontinuity and if it does not flow through it, whether or not the magnetic field penetrates it. The scheme is

Contact Discontinuity	$U_n = 0$,	$B_n \neq 0$
Tangential Discontinuity	$U_n = 0$,	$B_n = 0$
Shock Waves	$U_n \neq 0$	——
Parallel Shocks	"	$B_t = 0$
Perpendicular Shocks	"	$B_n = 0$
Oblique Shocks (Fast, Slow, Intermediate)		$B_t \neq 0$, $B_n \neq 0$

Contact Discontinuities: If the no-flow condition $U_n = 0$ is imposed on the six continuity relations, one quickly finds that all of the listed parameters are continuous across the discontinuity except density, ρ, which is left unspecified. Thus the density may change across a contact discontinuity, but since the pressures ($p_\parallel$ and $p_\perp$, or p in the case of isotropic pressure) are continuous, the temperature must change also to maintain the pressure constant. Since a discontinuity in temperature should rapidly be dispersed by heat flux parallel to the magnetic field (recall $B_n \neq 0$), such a discontinuity is not expected to occur in solar system plasmas except possibly for short intervals of time.

Tangential Discontinuities: The dual imposition of the conditions of no cross flow ($U_n=0$) and no field penetration ($B_n=0$) leads to one non-trivial continuity relation

$$[[p_\perp + \frac{B_t^2}{2\mu_o}]] = 0 \qquad \text{(III.66)}$$

This equation expresses the condition of static pressure balance in the direction normal to the discontinuity. Thus $p_\perp$ may change across the discontinuity, but $B_t^2/2\mu_o$ must change to maintain constant total static pressure. This type of discontinuity appears to be relatively common in solar system plasmas.

Shock Waves 1: Ordinary (non-MHD) Shock Waves:

It is helpful to begin a discussion of MHD shock waves with a review of ordinary gas dynamic shocks. Since the pressure in this case is a scalar, we may make the treatment more general by leaving the ratio of specific heats unspecified.

The continuity relations reduce to

$$[[\rho U_n]] \;=\; 0 \tag{III.67}$$

$$[[\rho U_n^2 + p]] \;=\; 0 \tag{III.68}$$

$$[[\rho U_n U_t]] \;=\; 0 \tag{III.69}$$

$$[[(\tfrac{1}{2}\rho U^2 + \tfrac{\gamma}{\gamma-1}p)U_n]] \;=\; 0 \tag{III.70}$$

Where the enthalpy $u+p = \gamma p/(\gamma-1)$. Combining (III.67) and (III.69) shows that $[[U_t]] = 0$, that is, the tangential component of the flow is continuous in gas dynamic shocks. We may therefore transform away the common U_t by a motion along the shock plane. This transformation does not affect any of the other quantities, but the index n on U_n may now be suppressed, since in the new frame $U_t = 0$.

Equations (III.67, 68 and 70) form a complete set for the downstream variables ρ_2, U_2 and p_2 if the upstream variables ρ_1, U_1, and p_1 are regarded as known. A single equation for U_2 is obtained by eliminating ρ_2 and p_2 between the three equations. This can be written in the form

$$U_2^2 - \frac{2\gamma}{\gamma+1}\left(1+\frac{1}{\gamma M_1^2}\right)U_1U_2 + \frac{\gamma-1}{\gamma+1}\left[1+\frac{2}{(\gamma-1)M_1^2}\right]U_1^2 = 0 \tag{III.71}$$

in which the upstream sonic Mach number, M_1, is defined as the ratio of the flow speed to the sound speed. It is given in general in terms of the other variables by

$$M^2 = \frac{\rho\, U^2}{\gamma\rho^2} \tag{III.72}$$

Now since the derivation of the continuity relations did not presume the existence of a discontinuity, they must also be consistent with the absence of a discontinuity. That is equation (III.71) must have $(U_2-U_1) = 0$ as one of its roots. Equation (III.71) can be factored into

$$(U_2-U_1)\left[U_2 - \frac{\gamma-1}{\gamma+1}\left(1 + \frac{2}{(\gamma-1)M_1^2}\right) U_1\right] = 0 \qquad \text{(III.73)}$$

The shock solution is obviously the second factor, which we write in terms of the downstream-to-upstream velocity ratio

$$\frac{U_2}{U_1} = \frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)M_1^2} \xrightarrow[M_1^2 \to \infty]{} \frac{\gamma-1}{\gamma+1} \qquad \text{(III.74)}$$

The density and pressure can now be found by substituting (III.74) back into the original shock equations

$$\frac{\rho_2}{\rho_1} = \frac{U_1}{U_2} = \frac{\gamma+1}{\gamma-1+\frac{2}{M_1^2}} \xrightarrow[M_1^2 \to \infty]{} \frac{\gamma+1}{\gamma-1} \qquad \text{(III.75)}$$

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1} \xrightarrow[M_1^2 \to \infty]{} \frac{2\gamma}{\gamma+1} M_1^2 \qquad \text{(III.76)}$$

Note that in the limit of weak shocks ($M_1^2 \to 1$), the parameters become continuous across the shock. That is, the shock disappears.

The shock waves that occur in solar system plasmas are in many instances strong shocks that satisfy the condition of the <u>hypersonic limit</u>, $M_1^2 >> 1$. The bow shocks of planetary magnetospheres and flare-driven solar blast waves are examples of strong shock waves that occur in the solar wind. The shock solutions have the interesting property of predicting an asymptotic limit on the degree of compression that a strong shock can produce. The hypersonic limits to the density and velocity ratios are given explicitly in (III.74, 75). The hypersonic limit of the post-shock pressure is given in (III.76). If the physically correct value of γ ($\gamma = 5/3$) is used, the asymptotic limits to the ratios are $\rho_2/\rho_1 = 4$ and $U_2/U_1 = 1/4$. The gas can be compressed and slowed down by a factor of no more than four. The shock wave also heats the gas. The shock relations do not predict an upper limit to the degree of heating as the Mach number increases

$$\frac{k}{m} T_s = \frac{P_2}{\rho_2} \xrightarrow[M_1^2 \to \infty]{} \frac{\gamma-1}{(\gamma+1)^2} M_1^2 U_1^2 \qquad \text{(III.77)}$$

The temperature of the solar wind gas heated by bow shocks and blast waves can be more than an order of magnitude greater than the pre-shock temperature.

In gas dynamics shock waves play the role of converting supersonic (M>1) flows to subsonic (M<1) in situations where the M<1 condition is needed to communicate to the fluid by means of pressure waves information about externally imposed constraints so that the fluid can adjust to them. The flow of a supersonic gas around a blunt body (e.g. the solar wind around a planetary magnetosphere), requires the interposition of a standing shock wave (usually detached from the body) to allow the gas to flow around the body. We can demonstrate directly from the shock solutions that the downstream sonic Mach numer is less than or equal to unity.

$$M_2^2 = \frac{\rho_2 V_2^2}{\gamma p_2} = \frac{(\gamma-1)M_1^2 + 2}{2\gamma M_1^2 - (\gamma-1)} \xrightarrow[M_1^2 \to \infty]{} \frac{\gamma-1}{2\gamma} \qquad \text{(III.78)}$$

Note that in the limit of weak shocks $M_1^2 = 1$, (III.78) gives $M_2^2 = 1$, that is, there is no change, as expected. Differentiation of (III.78) shows that dM_2^2/dM_1^2 is a negative definite quantity

$$\frac{dM_2^2}{dM_1^2} = -\left[\frac{\gamma+1}{2\gamma M_1^2-(\gamma-1)}\right]^2 < 0 \qquad \text{(III.79)}$$

Hence $M_2^2 < 1$ for $M_1^2 > 1$. In the hypersonic extreme, M_2^2 approaches a limiting value, which for $\gamma = 5/3$ is $M_2^2 \to 1/5$.

A fundamental property of shock waves is their adiabatic character. They dissipate some of the flow energy and convert it to heat, thereby raising the specific entropy of the gas. This statement can be verified directly by calculating the change in the adiabatic constant α of eq. (I.82).

$$\frac{\alpha_2}{\alpha_1} = \frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^{\gamma} = \left(\frac{1}{\gamma+1}\right)^{\gamma+1} [2\gamma M_1^2 - (\gamma-1)] \left(\gamma-1 + \frac{2}{M_1^2}\right)^{\gamma} \qquad \text{(III.80)}$$

The relation reduces to $\alpha_2 = \alpha_1$ when $M_1^2 = 1$. The trend in the change in α_2/α_1 for higher Mach numbers is given by

$$\frac{d}{dM_1^2}\left(\frac{\alpha_2}{\alpha_1}\right) = \left(\frac{1}{\gamma+1}\right)^{\gamma+1} \frac{2\gamma(\gamma-1)\left(\gamma-1 + \frac{2}{M_1^2}\right)^{\gamma-1}}{M_1^4} (M_1^2-1)^2 \qquad \text{(III.81)}$$

Because of the $(M_1^2-1)^2$ term, α_2/α_1 changes very slowly in the low Mach number range ($M_1^2 \gtrsim 1$). For $M_1^2 \neq 1$ the gradient is always positive, implying that $\alpha_2 > \alpha_1$ when $M_1^2 > 1$.

<u>Shock Waves 2: Parallel Shocks</u> We return now to the discussion of shock waves in magnetized plasmas. The dissipative nature of the shock tends to change the components of the anisotropic pressure in ways that are not predicted by the continuity relations (recall that there is one more unknown than there are equations if $p_\parallel$ and $p_\perp$ are retained). Without additional information, it is not possible to learn how $p_\parallel$ and $p_\perp$ separately change across a shock wave. Therefore, we must sacrifice this level of detail, and represent the pressure again by its isotropic form. In this case we may also leave γ arbitrary.

A parallel MHD shock wave is characterized by the condition $\vec{B}||\hat{n}$, or $B_t = 0$. When this condition is imposed on the MHD continuity relations one finds that they reduce identically to the ordinary gas dynamic shock relations that were reviewed under the previous heading (that is, eqs. III.67 through 70). The magnetic field strength disappears as an explicit variable except in Equation (III.59) which since $B_n = B$ becomes in this instance

$$[[B]] = 0 \qquad \text{(III.82)}$$

The field strength is continuous across the shock.

The treatment for the non-magnetic variables is identical to that developed for the gas dynamic case and all of the solutions given there apply in this situation also. In particular we were allowed to transform to a frame of reference in which $U_t = 0$. In this frame then $\vec{U}||\vec{B}$ and we have prescribed the condition necessary for the existence of an equipotential domain (see Section II.4). Thus there exists a streamline constant κ such that

$$\vec{B} = \kappa\rho\vec{U} \qquad \text{(III.83)}$$

It follows that since both B and ρU are conserved across the shock in this frame of reference

$$[[\kappa]] = 0 \qquad \text{(III.84)}$$

This result emphasizes the inert response of the magnetic field to its passage through a parallel shock.

<u>Shock Waves 3: Perpendicular Shocks</u>. Consider next as the opposite limit to the parallel shock, the perpendicular shock in which $\vec{B}\perp\hat{n}$, that is $B_n = 0$ and $B_t = B$. We note at the outset that the continuity relation for tangential momentum (III.63) again shows that U_t is continuous

across the shock, and hence as in the previous examples may be set equal to zero.

The continuity relations for the perpendicular shock can be written as

$$[[\rho U]] = 0 \tag{III.85}$$

$$[[\rho U^2 + p + \frac{B^2}{2\mu_o}]] = 0 \tag{III.86}$$

$$[[(\frac{1}{2}\rho U^2 + \frac{\gamma}{\gamma-1}p + \frac{B^2}{\mu_o})U]] = 0 \tag{III.87}$$

$$[[UB]] = 0 \tag{III.88}$$

The first and last of these can be combined to show that the field strength must change across the shock in the same proportion as does the density. That is

$$[[B/\rho]] = 0 \tag{III.89}$$

As in the procedure adopted for treating the continuity relations of ordinary gas dynamic shocks, we assume the upstream parameters to be given, in this case including the field strength B_1, and solve for the downstream parameters. The elimination of ρ_2, p_2 and B_2 results in a cubic equation for U_2. The trivial solution $U_2 - U_1 = 0$ can be factored out leaving the quadratic equation

$$(\frac{U_2}{U_1})^2 - [\frac{\gamma-1}{\gamma+1} + \frac{2\gamma}{\gamma+1}(\frac{1}{\gamma S_1^2} + \frac{1}{2A_1^2})](\frac{U_2}{U_1}) - \frac{2-\gamma}{\gamma+1}\frac{1}{A_1^2} = 0 \tag{III.90}$$

In (III.90) the sonic Mach number is designated by the symbol S and A is the Alfvén Mach number

$$S^2 \equiv M^2 = \frac{\rho V^2}{\gamma p} \tag{III.91}$$

$$A^2 \equiv \frac{\rho V^2}{B^2/\mu_o} \tag{III.92}$$

One of the two solutions of the quadratic equation is non-physical in that it gives a negative value for U_2. The other solution gives the velocity ratio for a perpendicular MHD shock of arbitrary sonic and Alfvén Mach number. In the limit $B \to 0$, i.e. $A \to \infty$, equation (III.90) reduces to the corresponding equation for ordinary gas dynamic shocks (III.74). Therefore, in the dual hypersonic limit, $A^2 \to \infty$ and $S^2 \to \infty$, we recover the same limiting expression for U_2/U_1, ρ_2/ρ_1 and p_2/p_1. In particular in the dual hypersonic limit

$$\frac{\rho_2}{\rho_1} = \frac{B_2}{B_1} = \frac{U_1}{U_2} = \frac{\gamma+1}{\gamma-1} \qquad \text{(III.93)}$$

Thus in the case of a strong perpendicular shock such as a planetary bow shock or a solar blast wave (for both of which $\gamma = 5/3$), the density and the field strength jump by a factor of four, and the velocity (in the reference frame of the shock) drops by the same factor.

Shock Waves 4: Oblique Intermediate Mode Shocks

For the case of oblique shocks ($B_{\shortparallel} \neq 0$, $B_t \neq 0$), there exist three types of MHD shocks that correspond to the three modes of small amplitude MHD waves. The two modes corresponding to the fast and slow mode waves are compressive. The mode corresponding to the intermediate mode wave is non-compressive, but if the pressure is anisotropic and $p_{\shortparallel}$ and $p_{\perp}$ change across the shock, the density will also change. (Recall that in the case of an anisotropic plasma, the continuity relations do not specify the change in $p_{\shortparallel}$ and $p_{\perp}$). We take up here the intermediate mode shock and retain the anisotropic form of the pressure tensor.

The distinction between intermediate shocks and the compressive shocks is revealed immediately by multiplying the continuity relation for tangential momentum (III.63) by the continuous variable $(B_n/\rho U_n)$ and adding to the result the continuity relation for tangential electric field (III.61). One obtains this way

$$[[(1 - \xi \frac{B_n^2}{\mu_o \rho U_n^2}) B_t U_n]] = 0 \qquad \text{(III.94)}$$

When this condition is satisfied by the vanishing of the term in parentheses, the shock described is the intermediate mode. The equation shows that when $(1 - \xi B_n^2/\mu_o \rho U_n^2) = 0$ on one side, it is zero on the other side as well. (The hybrid possibility that $B_t = 0$ on the other side, referred to as a switch-off shock, will not be considered. Such structures, if they exist, have not been demonstrated to play a significant role in solar system plasmas.) The term in parentheses is not zero in the compressive mode cases. Thus

$$1 - \xi \frac{B_n^2}{\mu_o \rho U_n^2} = 0 \text{ (intermediate mode shocks)} \qquad \text{(III.95)}$$

in which $\xi B_n^2/\mu_o \rho U_n^2$ may be evaluated on either side of the shock and thus this factor is continuous across an intermediate mode shock. Note that since B_n and ρU_n are continuous, it follows further that

$$[[\xi\rho]] = 0 \qquad \text{(III.96)}$$

Since the change in ξ can not be specified and ξ can in principle be different on the two sides, (III.96) shows that the density can in principle change across an intermediate mode shock in an anisotropic plasma. From this one sees that the same statement can be made concerning the propagation velocity of the shock, U_n, since ρU_n is continuous. The special but important case of an intermediate mode shock in an isotropic plasma should be noted separately

$$[[U_n]] = [[\rho]] = 0 \quad \text{(isotropic pressure, } \xi = 1) \qquad \text{(III.97)}$$

By use of the intermediate mode propagation equation (III.95), the change in the velocity vector can be found from (III.61 and 63). The result can be written in the form (Hudson, 1970)

$$[[\vec{U}]] = \left(\frac{\xi\rho}{\mu_o}\right)^{1/2} [[\frac{\vec{B}}{\rho}]] \qquad \text{(III.98)}$$

Equation (III.95) together with the continuity of B_n reduces the continuity relation for the normal component of momentum (III.62) to

$$[[p_{\perp} + \frac{B_t^2}{2\mu_o}]] = 0 \quad (\xi \text{ arbitrary}) \qquad \text{(III.99)}$$

It is worth noting that in the isotropic case (ξ=1) the continuity relation for energy for this mode simplifies to

$$[[p]] = 0 \qquad (\xi = 1) \qquad \text{(III.100)}$$

From which with (III.99) and $[[B_n]] = 0$ it follows that

$$[[B_t^2]] = [[B]] = 0 \quad (\xi = 1) \qquad \text{(III.101)}$$

Thus an intermediate mode shock wave propagating in an isotropic plasma is non-compressive and non-dissipative. It merely changes the directions of the magnetic field and the flow, while perserving their magnitudes.

Shock Waves 5: Oblique Fast and Slow Mode Shocks

The two compressive MHD shock modes have the following characteristics in common with the small amplitude fast and low modes waves.

The density increases across both of them, but in the fast mode shock the field strength increases while in the slow mode shock it decreases. Both types of shock can result from the non-linear evolution of finite amplitude fast and slow mode MHD waves. Thus, the relationship between the shocks and the waves is a direct heritage. We will consider only the case of isotropic pressure in the discussion of the compressive shocks.

Both types of compressive MHD shocks possess the notable property that the magnetic fields on the two sides on the shock and the normal to the shock are coplanar (that is $(\vec{B}_1 x \vec{B}_2)\cdot\hat{n} = 0$). This result, which is referred to as the coplanarity theorem, follows directly from (III.94) in which $\xi = 1$ and the factor in parentheses is not zero. Choose the tangential vector $\hat{t}$ to be perpendicular to $(\vec{B}_t)_1 \equiv \vec{B}_1 - B_n\hat{n}$. That is $\vec{B}_1 \cdot \hat{t} = 0$ for this choice of $\hat{t}$. But by (III.94) $\vec{B}_2 . \hat{t}$ is then zero also. Thus we have shown there is a choice of $\hat{t}$ which is perpendicular to $\vec{B}_1$ and $\vec{B}_2$, and by definition it is also perpendicular to $\hat{n}$. Hence $\vec{B}_1$, $\vec{B}_2$ and $\hat{n}$ are coplanar.

The coplanarity theorem has found use in the analysis of data on shock waves obtained from measurements in space. The orientation of the shock surface as given by its normal $\hat{n}$ can be obtained from a measurement of the magnetic fields on the two sides of the shock by the following procedure. Since B_n is continuous, $\vec{B}_1 - \vec{B}_2$ contains no normal component. Thus $\vec{B}_1 - \vec{B}_2$ lies in the plane of the shock. By the coplanarity theorem, $\vec{B}_1 x \vec{B}_2$ also lies in the plane of the shock. Hence

$$\hat{n} \,||\, (B_1 - B_2) \times (B_1 x B_2) \qquad \text{(III.102)}$$

It is not useful in the case of compressive oblique MHD shocks to proceed as in the previous examples of eliminating all but one of the downstream variables to arrive at a single equation for, say, $(U_n)_2$. The resulting equation is fifth order, and become fourth order after the trivial solution $(U_n)_2 - (U_n)_1 = 0$ has been factored out. However, an important ordering of the post-shock normal flow velocities can be stated on physical grounds. In order to avoid the non-physical possibility of an intermediate mode shock catching up to a preceding fast mode shock or a slow mode shock catching up to a preceeding intermediate mode shock, the post-shock speeds must be ordered according to

$$(\rho U_n{}^2)_2 \text{ (fast)} > \frac{B_n{}^2}{\mu_o} > (\rho U_n)_1 \text{ (slow)} \qquad \text{(III.103)}$$

in which $B_n 2/\mu_o \rho$ is the propagation speed of an intermediate mode shock in an isotropic gas (eq. III.95). This ordering determines whether the field strength increases or decreases across the shock, as can be seen by multiplying (III.94) by the continuous variable ρU_n

$$[[(\rho U_n^{\ 2} - \frac{B_n^{\ 2}}{\mu_o})B_t]] = 0 \qquad \text{(III.104)}$$

and solve for $(B_t)_2$ in terms of $(B_t)_1$

$$(B_t)_2 = \frac{(\rho U_n^{\ 2})_1 - \frac{B_n^2}{\mu_o}}{(\rho U_n^{\ 2})_2 - \frac{B_n^2}{\mu_o}} (B_t)_1 \qquad \text{(III.105)}$$

Consider first the case of a fast mode shock. Then both $(\rho U_n^2)_1 > B_n^2/\mu_o$ and $(\rho U_n^{\ 2})_2 > B_n^2/\mu_o$, the first because the upstream flow speed must be super-Alfvenic in order to have a fast shock at all, and the second by (III.103). But $(\rho U_n^2)_1 > (\rho U_n^{\ 2})_2$ since by factoring out the continuous term ρU_n, this becomes $(U_n)_1 > (U_n)_2$, which is guaranteed by the compressive nature of the shock. In summary, then both the numerator and the denominator in (III.104) are positive, and the numerator is larger than the denominator. Hence

$$(B_t)_2 > (B_t)_1 \quad \text{(fast mode shock)} \qquad \text{(III.106)}$$

and since $(B_n)_2 = (B_n)_1$, it also follows that

$$B_2 > B_1 \quad \text{(fast mode shock)} \qquad \text{(III.107)}$$

In the case of a slow mode shock, both the numerator and denominator are negative, the first because of (III.103) and the second because $(U_n)_2$ must be even smaller than $(U_n)_1$ by the compressive nature of the shock. The latter condition also means that the magnitude of the denominator is greater than the magnitude of the numerator. Hence

$$(B_t)_2 < (B_t)_1 \quad \text{(slow mode shock)} \qquad \text{(III.108)}$$

and again since $(B_n)_2 = (B_n)_1$

$$B_2 < B_1 \quad \text{(slow mode shock)} \qquad \text{(III.109)}$$

Figure III.4 shows the difference in the magnetic signatures of fast and slow mode shocks. On the downstream side of a fast mode shock the field bends away from the shock normal, giving rise to a compression of flux density, which is the same as an increase in field strength. On the downstream side of a slow mode shock, the field bends toward the shock normal resulting in a reduction of field strength.

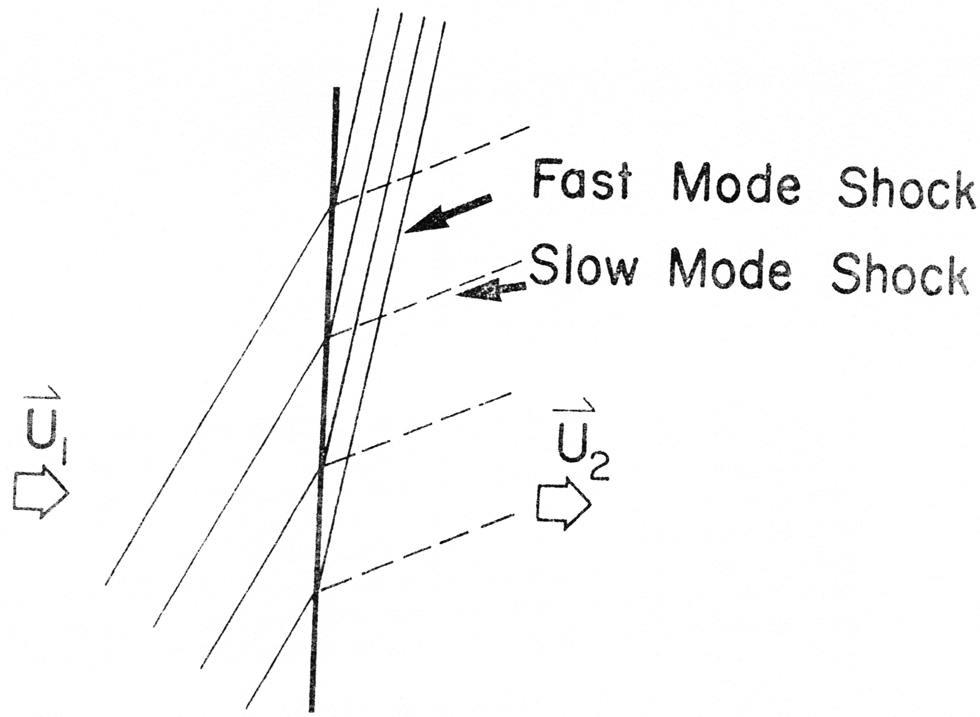

Figure III.4 Magnetic signatures of fast and slow mode shock waves

All three MHD shock modes have been observed in solar system plasmas, especially in the solar wind. Slow mode shocks play an expecially important role in some models of magnetic merging (Vasyliunas, 1975).

IV. MHD INSTABILITIES

This section reviews the four basic instabilities to which MHD fluids are subject. Two of the four are common to all fluids, namely the Kelvin-Helmholtz instability, colloqually known as the "wind over water" instability, and the Rayleigh-Taylor instability, which is called the flute instability in plasma physics and the interchange instability in magnetospheric physics. The other two, the firehose instability and the mirror instability, are caused by differences between $p_{\parallel}$ and $p_{\perp}$ in an anisotropic magnetized plasma, and are therefore peculiar to collisionless MHD fluids. We begin with instabilities driven by pressure anisotropy.

IV.1 The Firehose and Mirror Instabilities

In the theory of small amplitude MHD waves in an anisotropic plasma, these instabilities present themselves in the form of nonpropagating, purely exponentially growing waves. The firehose instability is an exponential growth of the intermediate mode, and the mirror in-