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Three-fluid Ohm's law

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Abstract. We develop a three-fluid theory, including electrons, ions, and neutral particles and their collisions, to describe the interaction between the magnetosphere, ionosphere, and thermosphere. We derive Ohm's law including the collisions between ions, electrons, and neutrals. In particular, our analysis includes the effects of electron-ion collisions. The electron and ion equations are solved in a completely coupled manner. The complete form of three-fluid Ohm's law in the plasma frame is obtained for a steady-state, uniform, three-fluid flow. This form of Ohm's law can be simplified to obtain the ideal MHD frozen-in condition when all collisions are neglected and the generalized Ohm's law when the collisions with neutrals are neglected. In our form the neutral velocity in the conventional collisional Ohm's law is replaced by the plasma velocity. More physical insights can be obtained from this form of Ohm's law in the context of magnetosphere-ionosphere coupling. This form describes the magnetosphere-ionosphere coupling without involvement of the neutral wind. The ionospheric velocity continuously deviates from the electric drift velocity from high altitudes to low altitudes. In addition, the plasma momentum equation for steady-state uniform three-fluid flow is derived. It describes ionosphere-thermosphere coupling. The Ohm's law in the neutral wind frame, or the conventional form of Ohm's law, can be derived by combining the Ohm's law in the plasma frame with the momentum equation. The conductivities become the same as the conventional ones under a few approximations. Combination of the two forms of Ohm's law provides the relationship between the electric field and the ionospheric velocity. When the neutral wind stays still, this relationship can be used to describe magnetosphere-ionosphere coupling.

1. Introduction

The coupling between the magnetosphere, ionosphere, and thermosphere occurs in three distinct plasma regimes ranging from highly ionized collisionless plasmas to weakly ionized collisional particles. Each of these regimes can be described by its own set of motion equations. The coupling between regions is determined by forces associated with the electromagnetic field, the electric currents, and interparticle collisions, which appear commonly among these equation sets.

In high-altitudes in the magnetosphere, say, >400 km, charged particles can be treated as collisionless. Collisionless magnetohydrodynamics (MHD) or ideal MHD is often used to describe the processes in this region. In this region the magnetic field can be described as being "frozen-in" to the plasma. When the plasma moves at speed $\mathbf{V}$ with respect to an observer, an electric field $\mathbf{E} = -\mathbf{V} \times \mathbf{B}$ is observed, where $\mathbf{B}$ is the magnetic field. The frozen-in condition is a form of Ohm's law when the conductivity goes to infinity. Under this condition the electric field and electric current are decoupled. The current is derived from Ampere's law. Since collisions essentially do not exist, the coupling with the ionosphere is performed via the electromagnetic field and currents. In this approach, electrons and ions are treated as a single fluid.

In the upper ionosphere, the *F*-region, for example, < 400 km, particles remain ionized. However, because the density of the plasma is higher, collisions between electrons and ions may not be negligible; see Figure 1, for example. The electrons and ions are strongly coupled but have to be treated separately. Under this condition the ions and electrons can be described in a mutual frame of reference of the plasma and the interaction between the two can be described by the so-called generalized Ohm's law [e.g., *Gombosi*, 1998]. This theory is based on a two-fluid treatment. The most important terms in the generalized Ohm's law relevant to our study are $\mathbf{j} = \sigma'_{\parallel} E_{\parallel} + \sigma'_{\mathrm{P}}(\mathbf{E}_{\perp} + \mathbf{U} \times \mathbf{B}) + \sigma'_{\mathrm{H}} \mathbf{b} \times (\mathbf{E} + \mathbf{U} \times \mathbf{B})$, where $\mathbf{U}$ is the velocity of the plasma (with respect to the observer) and $\sigma'_{\parallel}$, σ'_{P}, and σ'_{H} are the parallel, Pedersen, and Hall conductivities for the generalized Ohm's law, respec-

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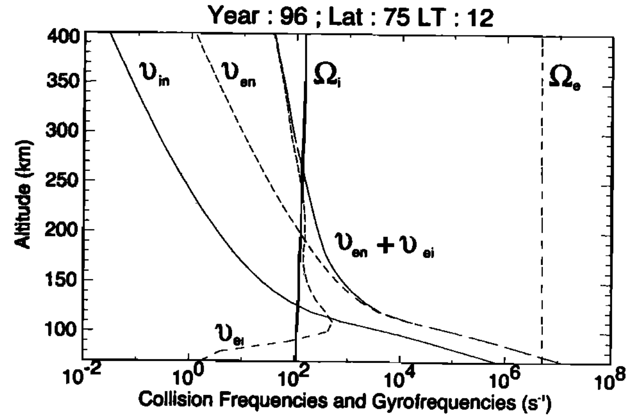

Figure 1. Collision frequencies and gyrofrequencies as functions of altitude at local noon at 75° latitude. They are determined based on the observations/laboratory experiments and the formula of *Kelley* [1989]. The anisotropy in the collision frequencies become unimportant above 80 km. More detailed discussion on the relationships among these quantities and anisotropies is given by *Richmond* [1995].

tively. It is important to point out that the conductivities in the generalized Ohm's law depend on electron-ion collision frequency ν_{ei}, and do not depend on neutral collision. The parallel conductivity goes to infinity, while the Pedersen conductivity goes to zero in nearly collisionless regimes, in contrast to ideal MHD, in which all conductivities go to infinity. In this region the electric field, plasma velocity, and current are related through the generalized Ohm's law, but they cannot be fully determined unless other relations are incorporated. The coupling with other regions is through the electromagnetic field and currents that are normal to the boundaries between regions.

In the *E*-region ionosphere, typically considered to be below ~ 150 km, the collisions between neutrals and charged particles become important and may even be dominant. The charged particles are coupled with the upper ionosphere via electromagnetic field and currents, and the neutral particles in the thermosphere are coupled with the charged particles through collisions. This region can be described as a three-fluid system. The coupling is expressed by a conventional collisional Ohm's law $\mathbf{j} = \sigma''_\parallel E_\parallel + \sigma''_P(\mathbf{E}_\perp + \mathbf{u}_n \times \mathbf{B}) + \sigma''_H \mathbf{b} \times (\mathbf{E} + \mathbf{u}_n \times \mathbf{B})$ [e.g., *Kelley*, 1989; *Richmond*, 1995; *Luhmann*, 1995, and references herein], where $\mathbf{u}_n$ is the velocity of the neutral wind, and $\sigma''_\parallel, \sigma''_P$ and σ''_H are the parallel, Pedersen, and Hall conductivities for the conventional Ohm's law, respectively. We note here that many of the commonly used expressions for the conductivities are derived when the electron-ion collisions are ignored [e.g., *Luhmann*, 1995]. The conductivities are typically expressed in terms of the ion-neutral collision and electron-neutral collision frequencies, in contrast to those in the generalized Ohm's law. *Kelley* [1989, p. 38] commented that his inclusion of the electron-ion collisions in some expressions of conductivities are not self-consistent. *Richmond* [1995] self-consistently included electron-ion collisions in the parallel conductivity. The most important difference from the generalized Ohm's law is the replacement of the plasma velocity with the velocity of neutral particles. This difference can be understood, as each form is written in a different frame of reference. An immediate consequence of this difference is that the ionospheric plasma velocity is not explicitly specified. Nevertheless, in order to study magnetosphere-ionosphere coupling, the difference between the magnetospheric convection and ionospheric velocity, instead of the difference between the magnetospheric convection and the neutral velocity, is one of the most important pieces of information, and it is not explicitly specified in the conventional Ohm's law.

In section 2, we present a three-fluid treatment to derive a general expression of Ohm's law that includes electron-ion, electron-neutral, and ion-neutral collisions. This form of Ohm's law is cast in the plasma frame, different from the conventional form used in the magnetosphere-ionosphere-thermosphere coupling as discussed above. It is the same as the generalized Ohm's law, although the conductivities are evaluated differently because of the inclusion of neutral-plasma collisions. This form can easily retrieve the generalized Ohm's law when neutral collision frequencies go to zero and the frozen-in condition when all collision frequencies are zero. In addition, the momentum equation for steady-state, uniform three-fluid flow is also derived. In section 3 the simplification of the conductivities in each region of the Earth's environment is discussed. The complete form of the conductivities for the Ohm's law in the neutral wind frame appears too complicated for any practical use. Approximations have to be made to derive the conventional conductivities. The approximations and mathematical details to derive the conventional conductivities are presented in section 4. Some of the physical consequences of Ohm's law and the momentum equation in terms of magnetosphere-ionosphere-thermosphere coupling are briefly discussed in section 5. More complete discussion will be presented in a later work.

2. Three-Fluid Treatment: Momentum Equation and Ohm's Law in Plasma Frame

A full treatment of three-species, i.e., electrons, ions, and neutrals, can be started from the kinetic equations for each species. One then integrates these equations over the phase space, defines macroscopic quantities, and derives various moment equations for each species [e.g., *Gombosi*, 1994]. These moment equations and macroscopic quantities describe each species as a fluid without invoking the motion of each individual particle. These equations can include the interaction or collisions among different species. In this study we focus on the behavior of the momentum equation for each of the three species. These equations are, in the Earth's frame, for ions,

$$N_e m_i \frac{D\mathbf{u}_i}{Dt} = -\nabla \mathbf{P}_i + eN_e(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) + \mathbf{F}_i - N_e m_i \nu_{in}(\mathbf{u}_i - \mathbf{u}_n) - N_e m_i \nu_{ie}(\mathbf{u}_i - \mathbf{u}_e), \quad (1)$$

for electrons,

$$N_e m_e \frac{D\mathbf{u}_e}{Dt} = -\nabla \mathbf{P}_e - eN_e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + \mathbf{F}_e - N_e m_e \nu_{en}(\mathbf{u}_e - \mathbf{u}_n) - N_e m_e \nu_{ei}(\mathbf{u}_e - \mathbf{u}_i), \quad (2)$$

and for neutrals,

$$N_n m_n \frac{D\mathbf{u}_n}{Dt} = -\nabla \mathbf{P}_n + \mathbf{F}_n - N_n m_n \nu_{ni}(\mathbf{u}_n - \mathbf{u}_i) - N_n m_n \nu_{ne}(\mathbf{u}_n - \mathbf{u}_e), \quad (3)$$

where e, N_e, $\mathbf{P}_\beta$, $\mathbf{E}$, $\mathbf{u}_\beta$, $\mathbf{B}$, m_β, N_n, and $\nu_{\beta\gamma}$ are electric charge of a particle, number density of each charged species, pressure tensor of particle type β, electric field, velocity of particle type β, magnetic field, mass of particle β, number density of the neutrals, and collision frequency between particle types β and γ, respectively. Subscripts i, e, and n denote ions, electrons, and neutrals, respectively. Vector $\mathbf{F}_\beta$ denotes other forces, such as gravity, centrifugal force, and Coriolis force, exerting on species β. We have assumed that the ions are singly charged and that charge neutrality holds in the plasma. We have neglected possible wave-particle collisions. To completely solve the motion of each species or processes in each region in space, other moment equations and Maxwell equations are required to close the equation sets. Ohm's law and the momentum equation to be shown later can be used to replace the electron and ion momentum equations. The focus of our study is to examine the motion of charged particles and their effects on the neutrals, and therefore we will only briefly discuss (3) in section 5. The effects of neutrals on charged particles are included in the neutral collision terms in (1) and (2). As explained in the Introduction, the coupling between regions is accomplished via the electromagnetic field, electric currents and collisions between species. Therefore, in the main body of the discussion, when discussing couplings, we neglect the effects from total time derivatives, pressure gradient forces, and additional forces. The effects of these terms will be discussed in a separate work. In this paper the sum of these terms will be referred to as "other forcings".

The three-fluid force balance equations for ions and electrons in the main body of our discussion are

$$eN_e(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) = N_e m_i \nu_{in}(\mathbf{u}_i - \mathbf{u}_n) + N_e m_i \nu_{ie}(\mathbf{u}_i - \mathbf{u}_e), \quad (4)$$

$$-eN_e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = N_e m_e \nu_{en}(\mathbf{u}_e - \mathbf{u}_n) - N_e m_e \nu_{ei}(\mathbf{u}_i - \mathbf{u}_e). \quad (5)$$

It is important to point out that the two equations are coupled by the electron-ion collision terms. Without these terms, (4) does not contain the electron velocity and (5) contains no ion velocity. In this case the two equations can be solved separately. The inclusion of the electron-ion collision terms couples the two equations, leading to mathematical complicity in finding the solutions. As can be seen in Figure 1, the electron-ion collision frequency is greater than the electron-neutral collision frequency above 190 km. The electron-ion collision term in (2) cannot be readily neglected. In this study we include the electron-ion collisions throughout the mathematical procedure.

We define the electric current density as

$$\mathbf{j} = eN_e(\mathbf{u}_i - \mathbf{u}_e), \quad (6)$$

and the velocity of the plasma flow as

$$\mathbf{U} = \frac{m_i \mathbf{u}_i + m_e \mathbf{u}_e}{m_i + m_e}. \quad (7)$$

Here we emphasize that the plasma velocity is defined in an inertial frame of reference, such as the Earth's frame. Noticing $m_i \gg m_e$, the plasma flow velocity becomes

$$\mathbf{U} = \mathbf{u}_i + \frac{m_e}{m_i}\mathbf{u}_e. \quad (8)$$

If the velocities for ions and electrons are of the same order, the plasma flow is in the ion flow direction. However, it is not necessarily true that the electric current is in the same direction as the ion flow direction. The corresponding ion and electron velocities are

$$\mathbf{u}_i = \mathbf{U} + \frac{m_e}{m_i}\frac{1}{eN_e}\mathbf{j}, \quad (9)$$

$$\mathbf{u}_e = \mathbf{U} - \frac{1}{eN_e}\mathbf{j}, \quad (10)$$

where a term of m_e/m_i in the electric current in (10) has been neglected. Substituting (9) and (10) into (4) and (5), while noting $m_i N_e \nu_{ie} = m_e N_e \nu_{ei}$, yields

$$eN_e(\mathbf{E} + \mathbf{U} \times \mathbf{B}) + \frac{m_e}{m_i}\mathbf{j} \times \mathbf{B} = N_e m_i \nu_{in}(\mathbf{U} - \mathbf{u}_n) + \frac{m_e}{e}(\nu_{in} + \nu_{ei})\mathbf{j}, \quad (11)$$

$$-eN_e(\mathbf{E} + \mathbf{U} \times \mathbf{B}) + \mathbf{j} \times \mathbf{B} = N_e m_e \nu_{en}(\mathbf{U} - \mathbf{u}_n) - \frac{m_e}{e}(\nu_{en} + \nu_{ei})\mathbf{j}. \quad (12)$$

Here we should point out that the $\mathbf{j} \times \mathbf{B}$ (j-cross-B) force is more important in affecting electron flow than in affecting ion flow, as seen by the factor of m_e/m_i in the $\mathbf{j} \times \mathbf{B}$ force term in (11). Mathematically, $\mathbf{u}_i$ and $\mathbf{u}_e$ in (4) and (5) are now replaced by $\mathbf{U}$ and $\mathbf{j}$ in (11) and (12). Physically, the electron and ion flows are now described in a single frame of reference of the plasma flow and their relative motion is described by the current. However, the physical meaning of each of (11) and (12) becomes less clear.

In principle, one equation can be added to or subtracted from the other. The summation reveals the effects that are dependent on the type of the electric charges, while the subtraction enhances the effects that are independent on the type of charges. We first add the two equations and obtain

$$\mathbf{j} \times \mathbf{B} = N_e(m_i \nu_{in} + m_e \nu_{en})(\mathbf{U} - \mathbf{u}_n) + \frac{m_e}{e}(\nu_{in} - \nu_{en})\mathbf{j}. \quad (13)$$

The equation is the momentum equation for the plasma. An important point is that electron-ion Coulomb collisions do not affect the plasma flow. In the direction perpendicular to the current, when other forcings are not considered, the $\mathbf{j} \times \mathbf{B}$ force is needed to overcome the collisions of the plasma with neutrals. The last term is a frictional force along the electric current due to the difference in neutral-charged particle collisions. It is of the order of the frequency of neutral-charged particle collisions divided by the electron gyrofrequency and is generally small above 80 km.

When subtracting (11) from (12), there is a possibility of eliminating one of the variables. One has the choice to eliminate either the plasma velocity $\mathbf{U}$ or the neutral velocity $\mathbf{u}_n$. Each leads to a form of Ohm's law. However, the physical insights that can be gained from the examination of each are different. The two forms are linked by the momentum equation (13). Conventional theories for magnetosphere-ionosphere-thermosphere coupling have chosen to eliminate $\mathbf{U}$, while we proceed with the elimination of the $\mathbf{u}_n$ terms in (11) and (12), and obtain

$$\begin{aligned} & eN_e(m_e\nu_{en} + m_i\nu_{in})(\mathbf{E} + \mathbf{U}\times\mathbf{B}) \\ +\ & \left(\frac{m_e}{m_i}m_e\nu_{en} - m_i\nu_{in}\right)\mathbf{j}\times\mathbf{B} \\ =\ & \frac{m_e}{e}\left[m_e\nu_{en}\nu_{ei} + m_i\nu_{in}(\nu_{en} + \nu_{ei})\right]\mathbf{j}, \end{aligned} \tag{14}$$

Equation (14) can be also written as

$$\mathbf{j} + \frac{\sigma_0}{eN_e}\alpha_H\mathbf{j}\times\mathbf{B} = \sigma_0\alpha_P(\mathbf{E} + \mathbf{U}\times\mathbf{B}), \tag{15}$$

where

$$\begin{aligned} \sigma_0 &= \frac{e^2N_e}{m_e\nu_{ei}}, \\ \alpha_P &= \frac{\nu_{ei}(m_i\nu_{in} + m_e\nu_{en})}{m_e\nu_{en}\nu_{ei} + m_i\nu_{in}(\nu_{en} + \nu_{ei})}, \\ \alpha_H &= \frac{\nu_{ei}\left(m_i\nu_{in} - \frac{m_e}{m_i}m_e\nu_{en}\right)}{m_e\nu_{en}\nu_{ei} + m_i\nu_{in}(\nu_{en} + \nu_{ei})}. \end{aligned} \tag{16}$$

When $\nu_{ei}, \nu_{in}, \nu_{en} \neq 0$, (15) can be solved for the current density to obtain Ohm's law:

$$\begin{aligned} \mathbf{j} &= \sigma_\parallel E_\parallel\mathbf{b} + \sigma_P(\mathbf{E}_\parallel + \mathbf{U}\times\mathbf{B}) \\ &\quad + \sigma_H\mathbf{b}\times(\mathbf{E} + \mathbf{U}\times\mathbf{B}) \\ &= \sigma_\parallel E_\parallel^*\mathbf{b} + \sigma_P\mathbf{E}_\perp^* + \sigma_P\mathbf{b}\times\mathbf{E}^*, \end{aligned} \tag{17}$$

where $\mathbf{b} = \mathbf{B}/B$ is the unit vector along the magnetic field, and $_\parallel$ and $_\perp$ are the parallel and perpendicular components, respectively, with respect to the magnetic field. The field-aligned, Pedersen, and Hall conductivities are given by

$$\begin{aligned} \sigma_\parallel &= \alpha_P\sigma_0, \\ \sigma_P &= \sigma_\parallel\frac{\nu_{ei}^2}{\nu_{ei}^2 + \alpha_H^2\Omega_e^2}, \\ \sigma_H &= \sigma_\parallel\frac{\alpha_H\nu_{ei}\Omega_e}{\nu_{ei}^2 + \alpha_H^2\Omega_e^2}, \end{aligned} \tag{18}$$

where $\Omega_e = eB/m_e$ is the electron gyrofrequency. The electric field $\mathbf{E}^* = (\mathbf{E} + \mathbf{U}\times\mathbf{B})$ is the electric field in the frame of reference moving with the ionospheric plasma. We should point out that the Pedersen and Hall conductivities referred to here are physically quite different from those used in the conventional collisional Ohm's law, which are double-primed in this paper and will be further discussed in section 4. The nomenclature we use here is standard when discussing the generalized Ohm's law [e.g., *Gombosi*, 1998].

3. Ohm's Law in Plasma Frame in the Earth's Environment

It should be pointed out that up to this point, no assumptions were made about the relative values among collision and gyro frequencies. The components of the conductivity tensor given by (18) are valid for a wide range of plasma conditions. In the following we discuss several more important parameter regimes. In the Earth's environment, these regimes can be described in terms of regions in space. However, we emphasize that for other planets, their satellites, comets, or other astronomical systems, the plasma conditions may be quite different from those discussed below, but similar analyses can be carried out. In our discussion, we refer to the last two terms in (17) as Pedersen and Hall effects, respectively. They are different from those as discussed in conventional theories because our Pedersen and Hall effects are described in the plasma frame. In high altitudes their functions, in fact, are interchanged with the conventional ones. Figure 1 shows the profiles of some collision frequencies as functions of height in the Earth's environment. In the following discussion they are used qualitatively to guide discussion. Each of the parameter regimes discussed is independent of these observations. In the Earth's environment, as shown in Figure 1, $m_i\nu_{in} \gg m_e\nu_{en}$. In fact, this condition is most likely to be true in a wide range of environments. This leads to $\alpha_P = \alpha_H = \nu_{ei}/(\nu_{en} + \nu_{ei})$. The conductivities can be simplified as

$$\begin{aligned} \sigma_\parallel &= \frac{eN_e\Omega_e}{B(\nu_{en} + \nu_{ei})}, \\ \sigma_P &= \frac{eN_e(\nu_{ei} + \nu_{en})\Omega_e}{B[\Omega_e^2 + (\nu_{ei} + \nu_{en})^2]}, \\ \sigma_H &= \frac{eN_e\Omega_e^2}{B[\Omega_e^2 + (\nu_{ei} + \nu_{en})^2]}. \end{aligned} \tag{18'}$$

The most important regions in the magnetosphere-ionosphere-thermosphere coupling are the following:

1. In high-altitudes in the magnetosphere, where all collisions can be neglected, i.e., $\nu_{ei}, \nu_{in}, \nu_{en} \sim 0$, we have $\sigma_\parallel \to \infty$ and $\alpha_P = \alpha_H \leq 1$. From (15), along the current direction, in order to keep the current finite, we have $\mathbf{E} = -\mathbf{U}\times\mathbf{B}$. Namely, with only the electric field drift, no current can be generated in a uniform plasma because the electrons and ions drift together. Therefore we have the collisionless MHD limit, or ideal MHD. Here we should point out that this result cannot be derived from (17), which is obtained assuming $\nu_{ei}, \nu_{in}, \nu_{en} \neq 0$. In ideal MHD the current can be generated by other forcings and nonuniformity. We should notice that there is a Hall effect in the direction perpendicular to the current. Namely, in the direction perpendicular to the current, the frozen-in condition does not perfectly hold unless the current is weak.

2. Above a few hundred kilometers in altitude (see Figure 1, for example), electron-neutral and ion-neutral collisions can be neglected. The electron-ion collisions become significant ($\nu_{en}, \nu_{in} \ll \nu_{ei}$), and the plasma is gyration dominated

$(\Omega_i, \Omega_e \gg \nu_{ei})$. Furthermore, $m_i\nu_{in} \gg m_e\nu_{en}$. In this limit we obtain $\alpha_P = \alpha_H = 1$ and

$$\begin{aligned} \sigma_\parallel &= \sigma_0 = \frac{eN_e\Omega_e}{B\nu_{ei}}, \\ \sigma_P &= \frac{eN_e}{B}\frac{\nu_{ei}}{\Omega_e}, \\ \sigma_H &= \frac{eN_e}{B}. \end{aligned} \tag{19}$$

This is the limit for ion-electron two-fluid (collisional) generalized Ohm's law when the effects of the pressure and inertial forces are not included. Collisions are due to Coulomb collisions between ions and electrons. We remark here that the conductivity along the magnetic field remains very high. The Hall conductivity is independent of the collision frequency and is proportional to the electron density. The Pedersen conductivity is proportional to the collision frequency. The Hall conductivity is much more important than the Pedersen conductivity in this region. Here we recall that our Hall and Pedersen conductivities are defined differently from the conventional ones. We will discuss this issue in sections 4 and 5.

3. At midaltitudes ($\sim$ 100 to 190 km), plasma collisions with neutral particles become important. The electron-neutral collision frequency ν_{en} becomes much larger than the electron-ion collision frequency ν_{ei} and the ion-neutral collision frequency ν_{in}, $(\nu_{en} \gg \nu_{in}, \nu_{ei})$. The ions become collision dominant, but the electrons remain gyration dominant, i.e., $\Omega_e \gg \nu_{ei}, \nu_{en}, \nu_{in}$. We have $\alpha_P = \alpha_H \sim \nu_{ei}/\nu_{en}$ and

$$\begin{aligned} \sigma_\parallel &= \alpha_P\sigma_0 = \frac{eN_e\Omega_e}{B\nu_{en}}, \\ \sigma_P &= \frac{eN_e}{B}\frac{\nu_{en}}{\Omega_e}, \\ \sigma_H &= \frac{eN_e}{B}. \end{aligned} \tag{20}$$

This is the most important transition region. The Hall conductivity remains more important than the Pedersen conductivity. Here we recall again that our Hall and Pedersen conductivities are defined differently from the conventional ones.

4. At low altitudes, below 80 km, neutral-charged particle collisions become dominant. In this region, $\nu_{en} \gg \nu_{ei}, \Omega_e$. Again, in this region, $m_i\nu_{in} \gg m_e\nu_{en}$ and $\alpha_P = \alpha_H = \nu_{ei}/\nu_{en}$. The conductivities in this region become

$$\begin{aligned} \sigma_\parallel &= \sigma_P = \frac{eN_e\Omega_e}{B\nu_{en}}, \\ \sigma_H &= \frac{eN_e}{B}\frac{\Omega_e^2}{\nu_{en}^2}. \end{aligned} \tag{21}$$

An important difference from the above regions is that the Pedersen conductivity becomes important or even dominant in this region. As the plasma density decreases with altitude in this region, the effects of electric field and current are diminishing with decreasing altitude.

It is interesting to note that at low altitudes the parallel conductivities are controlled by the electron-neutral collisions, $1/\nu_{en}$. At high-altitudes they are controlled by electron-ion collisions, $1/\nu_{ei}$. The Hall conductivity is controlled by the density profile in a large range of altitude above 100 km and then decreases toward lowest altitudes. As the relative importance of the gyromotion of charged particles decreases, the importance of the Pedersen conductivity increases as the altitude decreases. However, the Pedersen conductivity is also proportional to the plasma density. At very low altitudes the electron density becomes very low, and hence the value of the Pedersen conductivity becomes very small, although its importance increases compared with the Hall effect.

4. Ohm's Law in the Neutral Wind Frame: Comparison With Conventional Theory

In the discussion before introducing our (14), we noted the possibility of eliminating $\mathbf{U}$ in (11) and (12). The plasma velocity can also be eliminated by combining momentum equation (13) and Ohm's law (15). The actual resultant conductivities are quite complicated and have little value for practical application. In a wide range of environments, $m_e\nu_{en}$ is small when compared with $m_i\nu_{in}$. This yields

$$\begin{aligned} \mathbf{j} = \sigma''_\parallel E_\parallel &+ \sigma''_P(\mathbf{E}_\perp + \mathbf{u}_n \times \mathbf{B}) \\ &+ \sigma''_H \mathbf{b} \times (\mathbf{E} + \mathbf{u}_n \times \mathbf{B}), \end{aligned} \tag{22}$$

where the parallel conductivity is

$$\sigma''_\parallel = \frac{eN_e\Omega_e}{B(\nu_{en} + \nu_{ei})}, \tag{23}$$

the Pedersen conductivity is

$$\sigma''_P = \frac{eN_e}{B}\frac{\Omega_e\nu_{in}(\Omega_i\Omega_e + \nu_{en}\nu_{in} + \nu_{ei}\nu_{in})}{(\Omega_i\Omega_e + \nu_{en}\nu_{in} + \nu_{ei}\nu_{in})^2 + \nu_{in}^2\Omega_e^2}, \tag{24}$$

and the Hall conductivity is

$$\sigma''_H = \frac{eN_e}{B}\frac{\Omega_e^2\nu_{in}^2}{(\Omega_i\Omega_e + \nu_{en}\nu_{in} + \nu_{ei}\nu_{in})^2 + \nu_{in}^2\Omega_e^2}, \tag{25}$$

where $\Omega_i = eB/m_i$ is the ion gyrofrequency.

We recall here that the Pedersen and Hall conductivities with a double-prime are defined in the frame of reference of neutrals. They are physically different from those defined in (17). In comparison, the conventional three-fluid Ohm's law used in the magnetosphere-ionosphere interaction [e.g., *Luhmann*, 1995] is the same as (22). The parallel conductivity is

$$\sigma''_\parallel = \frac{eN_e}{B}\left(\frac{\Omega_e}{\nu_{en}} + \frac{\Omega_i}{\nu_{in}}\right), \tag{26}$$

the Pedersen conductivity is

$$\sigma''_P = \frac{eN_e}{B}\left[\frac{\Omega_e}{\nu_{en}}\left(\frac{\nu_{en}^2}{\nu_{en}^2 + \Omega_e^2}\right) + \frac{\Omega_i}{\nu_{in}}\left(\frac{\nu_{in}^2}{\nu_{in}^2 + \Omega_i^2}\right)\right], \tag{27}$$

and the Hall conductivity is

$$\sigma_{H}'' = \frac{eN_e}{B}\left[\frac{\Omega_e}{\nu_{en}}\left(\frac{\Omega_e\nu_{en}}{\nu_{en}^2+\Omega_e^2}\right) - \frac{\Omega_i}{\nu_{in}}\left(\frac{\Omega_i\nu_{in}}{\nu_{in}^2+\Omega_i^2}\right)\right]. \quad (28)$$

Equations (26) to (28) will be referred to as the conventional conductivities. Here we should emphasize that although (22) and conventional three-fluid Ohm's law are the same in form, there are differences in the actual evaluations of the conductivities. Equations (23) to (25) were derived under the assumption that $m_e\nu_{en}$ is small compared with $m_i\nu_{in}$. Although the electron-ion collision frequency is introduced to the conductivities by *Kelley* [1989], he noticed that his treatment is not self-consistent. *Richmond* [1995] included the electron-ion collisions in the parallel conductivity. Our parallel conductivity (23) is the same as that of *Kelley* [1989] and *Richmond* [1995] and as in (18′), indicating that the parallel conductivity in some forms of the conventional theory is accurate in the Earth's environment. Comparing (24) and (25) with (27) and (28), there are some apparent differences. First, in our results, as expected, there are terms involving the electron-ion collisions. Second, there are additional terms involving the gyrofrequencies. Finally, an interesting feature when comparing (24) and (25) with (27) and (28) is that the ion terms and electron terms are completely separated in the conventional conductivities but are coupled in our form. This can be understood by the assumption of decoupling the electron and ion equations when deriving them in the conventional theories. We do not make such an assumption, and hence the electron and ion motions are coupled.

Given $m_i\nu_{in} \gg m_e\nu_{en}$, in the Earth's environment, the following approximations are also true above 100 km, as shown in Figure 1: $\Omega_i\Omega_e \gg \nu_{en}\nu_{in}$, $\Omega_i\Omega_e \gg \nu_{ei}\nu_{in}$, and $\Omega_e \gg \nu_{en}$. Under these approximations, (24) and (25) are the Pedersen conductivity,

$$\sigma_P'' = \frac{eN_e}{B}\frac{\nu_{in}\Omega_i}{\Omega_i^2+\nu_{in}^2}, \quad (29)$$

and the Hall conductivity,

$$\sigma_H'' = \frac{eN_e}{B}\frac{\nu_{in}^2}{\Omega_i^2+\nu_{in}^2}. \quad (30)$$

With the same approximations, (27) and (28) become the same as (29) and (30).

Assuming $m_i\nu_{in} \gg m_e\nu_{en}$, the conventional parallel conductivity, (26) becomes

$$\sigma_{\parallel}'' = \frac{eN_e\Omega_e}{B\nu_{en}}. \quad (31)$$

When replacing ν_{en} with $\nu_e = \nu_{ei} + \nu_{en}$ in (30) [*Kelley*, 1989], (31) becomes the same as (23). Therefore the conventional conductivities are justified above 100 km.

For the other parameter regime, if $\nu_{en}\nu_{in} \gg \Omega_i\Omega_e, \nu_{ei}\nu_{in}$ and $\nu_{en} \gg \Omega_e$, we have

$$\sigma_P'' = \frac{eN_e\Omega_e\nu_{en}}{B[\Omega_e^2+\nu_{en}^2]}, \quad (29')$$

$$\sigma_H'' = \frac{eN_e\Omega_e^2}{B[\Omega_e^2+\nu_{en}^2]}. \quad (30')$$

Equations (29′) and (30′) are the same as (18′), since $\nu_{en} \gg \nu_{ei}$. Therefore, if the approximations to derive (29′) and (30′) are justified below 100 km, the conventional conductivities, in principle, are applicable in the Earth's environment under reasonable approximations.

Plasma environments in other places in space may be quite different. According to Table 9.2.1 of *Richmond* [1995], collision frequencies are proportional to the particle densities involved and electron collision frequencies are proportional to the electron temperature. The density and temperature as functions of height depend on the gravity, the composition, the energy sources of ionization, naming just a few, and other related processes. The gyrofrequencies, on the other hand, are proportional to the strength of the magnetic field. The approximations made in order to derive (29) and (30) may become poor in some other planets or their moons, near the Sun, in comets, and in other astronomical environments where the magnetic field is weaker, the density is higher, or electrons are hotter. In some of these environments, the effects of mass-loading and finite Larmor radius effects may become important.

The electron-ion collision frequency does not appear to play an important role in the conventional horizontal conductivities. This is because in the conventional form of Ohm's law, the plasma motion is not explicitly described. The effects of electron-ion collisions can be seen more clearly in the ion frame, or plasma frame, of reference.

5. Discussion

Let us first make the link between the mathematical terms and the physical processes. Our main mathematical results, i.e., equations (13), (17), and (18), hold for local and not global quantities. The link between the global quantities and the local quantities needs to be established separately. We apply our equations to the ionosphere-thermosphere regions. Here we recall that the ionosphere (thermosphere) refers to the charged (neutral) particles of the same region. The ionosphere is represented by **j**, **B**, **E**, **U**, and N_e, as well as electron-ion collision frequency ν_{ei}. The thermosphere is represented by $\mathbf{u}_n$. The coupling between the ionosphere and thermosphere is indicated by the neutral-plasma collision frequencies ν_{en} and ν_{in}. The magnetosphere near the ionosphere can be considered as collision free. It is coupled with the ionosphere by **B** and **E**, which are shared by both the ionosphere and magnetosphere. The magnetospheric electric field and the ionospheric electric field are directly related. Therefore (17) describes the coupling between the ionosphere and magnetosphere. The effects of the thermosphere are partially included by neutral collisions, but the neutral wind flow effects are not included. Likewise, (13) provides the coupling relationship between the ionosphere and thermosphere. Here we note again that **U** is the plasma flow velocity in the ionosphere. It is not defined in the neutral wind frame. From (9) and (10) the plasma flow velocity is similar to the ion velocity. Notice that the electric current is independent of frame of reference in nonrelativistic flow. Electrons move antiparallel to the electric current in the plasma frame of reference.

Let $\mathbf{V}_{\mathrm{drift}} = -\mathbf{E} \times \mathbf{B}/B^2$ be the electric drift velocity. The difference between the ionospheric velocity and the electric drift velocity is

$$\mathbf{U} - \mathbf{V}_{\mathrm{drift}} = -\frac{\mathbf{j} \times \mathbf{B}}{\sigma_0 \alpha_{\mathrm{P}} B^2} + \frac{\alpha_{\mathrm{H}}}{e N_{\mathrm{e}} \alpha_{\mathrm{P}}} \mathbf{j}. \tag{32}$$

The ratio of the last two terms in (32) is $(\nu_{\mathrm{ei}} + \nu_{\mathrm{en}})/\Omega_{\mathrm{e}}$, which is small above 80 km.

It is important to estimate the magnitude of the difference between the two velocities. Since our current is defined in the Earth's frame, the current in (32) can be estimated from (22). The velocity difference is of the order of $\Omega_{\mathrm{i}}\nu_{\mathrm{in}}/(\Omega_{\mathrm{i}}^2 + \nu_{\mathrm{in}}^2)$ or $\nu_{\mathrm{in}}^2/(\Omega_{\mathrm{i}}^2 + \nu_{\mathrm{in}}^2)$, whichever is bigger, times V_{drift}, if the neutral wind stays still. The first factor is dominant at higher altitudes. It is peaked near 120 km with a value of 1/2 and equals the second factor at this height. The second factor continues to increase below 120 km and reaches a maximum value of 1.

Figure 2 summarizes the directions of involved vectors in the northern polar ionosphere for steady state southward interplanetary magnetic field (IMF) when the effects of the neutral wind velocity are not considered. We assume that the magnetospheric convection is antisunward and that the magnetospheric electric field $\mathbf{E}$ is duskward. The ionospheric electrons are frozen-in with the magnetospheric field, if the electron gyrofrequency is much larger than the electron collision frequencies, and hence $\mathbf{u}_{\mathrm{e}}$ is antisunward. The difference between the ionospheric ion and electron velocities is in the current direction. From Ohm's law in the neutral wind frame, current $\mathbf{j}$ is in the $\mathbf{E}$ direction above 120 km. Therefore the velocity difference is duskward above 120 km. This is consistent with that described in Figure 2.4 of *Kelley* [1989]. The ionospheric ion velocity is therefore antisunward and duskward. As the current increases from higher altitudes to lower altitudes, because of an increase in collisions, the ion velocity rotates more toward the dusk. The ionospheric plasma velocity $\mathbf{U}$ is the same as $\mathbf{u}_{\mathrm{i}}$. Because the electric field in the ionospheric plasma frame $\mathbf{E}^*$ is perpendicular to $\mathbf{j}$, it is in the antisunward direction when $\sigma_{\mathrm{H}} \gg \sigma_{\mathrm{P}}$ in equation (17).

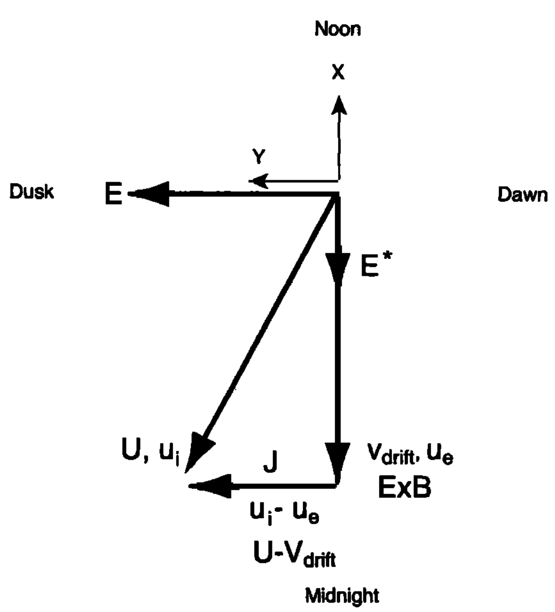

Figure 2. Illustration of the directions of the vectors in the northern polar cap for due southward interplanetary magnetic field. The magnetic field points into the page. The Sun is up in the x direction. The magnetospheric convection is antisunward, and the electric field is duskward. The plasma or ion velocity is antisunward and tilts toward dusk. The current is duskward.

The perpendicular conductivities defined in our Ohm's law are different from those defined in the conventional theory. The difference also leads to different physical understanding. Our conductivities are defined in the plasma frame. This frame moves with the ions. The current is carried by electrons in this frame. If there is no collision, the electrons conduct the electric drift motion moving with ions. Electrons stay still in the plasma frame. No current will appear. With collisions, electrons and ions move at different velocities. The separation of the two species creates an electric field in the plasma frame $\mathbf{E}^* = (\mathbf{E} + \mathbf{U} \times \mathbf{B})$. It is nearly perpendicular to the electric field in the inertial frame above 120 km. The electric drift motion produces the Hall current. Electron collisions make the electrons move preferably in the electric field direction while drifting, leading to the Pedersen current. Since in a large range of altitudes electrons are gyrodominant, the Hall current is much greater than the Pedersen current. This is also why the Hall conductivity in our definition is independent of collisions in this range.

The conductivities defined in the neutral wind frame mean something quite different. In the neutral wind frame the current can be carried by both electrons and ions. A significant component of the two motions is the same. It does not produce a net current. The net current in the electric field direction is produced by ion motion. This is the Pedersen current in conventional theory. Similarly, the electrons carry the Hall current. However, most of the electron current is cancelled by the ion drift at high altitudes.

When studying magnetosphere-ionosphere coupling, the difference between the magnetosphere and the ionosphere motions is most important. It is most useful to derive the relationship between the magnetospheric electric field and the ionospheric velocity. As we noted earlier, the electric current in our treatment is defined in the inertial frame. In fact, it is invariant of frame of reference in nonrelativistic flow. Combining equation (17) with (22) yields

$$\mathbf{U} = \frac{1}{B^2}\mathbf{E} \times \mathbf{B} + \frac{\sigma_{\mathrm{P}}''}{\sigma_{\mathrm{H}} B}\mathbf{E}, \tag{33}$$

where we have assumed a stay-still neutral wind and an altitude above 120 km. From this relationship one can see clearly that the ratio of ion motion in the electric field direction and the electric field drift direction is $\sigma_{\mathrm{P}}''/\sigma_{\mathrm{H}}$. For a given magnetospheric electric field the ionospheric convection velocity is determined.

The coupling between the ionosphere and thermosphere is primarily via neutral-charged particle collisions, because the electric current and electromagnetic field do not appear in (3). The rate of the momentum gain in neutrals from charged particles equals the momentum loss in the charged particles. From (1) to (3) we have

$$-N_n m_n[\nu_{ni}(\mathbf{u}_n - \mathbf{u}_i) + \nu_{ne}(\mathbf{u}_n - \mathbf{u}_e)] = N_e[m_i\nu_{in}(\mathbf{u}_i - \mathbf{u}_n) + m_e\nu_{en}(\mathbf{u}_e - \mathbf{u}_n)]. \quad (34)$$

From (13), the right-hand-side of (34) is

$$N_e[m_i\nu_{in}(\mathbf{u}_i - \mathbf{u}_n) + m_e\nu_{en}(\mathbf{u}_e - \mathbf{u}_n)] = N_e(m_i\nu_{in} + m_e\nu_{en})(\mathbf{U} - \mathbf{u}_n) + \frac{m_e}{e}(\nu_{in} - \nu_{en})\mathbf{j} = \mathbf{j} \times \mathbf{B}. \quad (35)$$

It is the gain in the momentum of the neutrals. In the regions without current, namely, $\mathbf{j} = 0$, the ionospheric plasma and the neutral wind move with the same speed. In other words, if the plasma and the neutral wind have different speeds, there exists a current, unless other forcings are present. If the neutral wind stays still, the neutral wind exerts a drag force on ions to balance the $\mathbf{j} \times \mathbf{B}$ force. In the low latitudes, ion drag opposes the neutral wind, especially during the daytime. In the polar cap, on the other hand, the solar wind supplies momentum to both the plasma and the neutrals in the same direction [*Kelley*, 1989, pp. 263-264].

To estimate the magnitude of the ionosphere-thermosphere coupling, we evaluate the order of the $\mathbf{j} \times \mathbf{B}$ force divided by $N_e m_i \nu_{in}$ in (35), which measures the difference between the ionospheric plasma and thermospheric neutral velocities. The frictional force, the last term in the middle expression, is small. From (32) the corresponding velocity of $\mathbf{j} \times \mathbf{B}/N_e m_i \nu_{in}$ is of the order of Ω_i/ν_{in} times the magnitude of $\mathbf{V}_{\text{drift}} - \mathbf{U}$. Since $\mathbf{V}_{\text{drift}} - \mathbf{U}$ is proportional to $\Omega_i\nu_{in}/(\Omega_i^2 + \nu_{in}^2)$ above 120 km and to $\nu_{in}^2/(\Omega_i^2 + \nu_{in}^2)$ below 120 km, the factor of the neutral wind and ionosphere velocity difference is $\Omega_i^2/(\Omega_i^2 + \nu_{in}^2)$ above 120 km and $\Omega_i\nu_{in}/(\Omega_i^2 + \nu_{in}^2)$ below 120 km. For a given electric field, namely, for a given $\mathbf{V}_{\text{drift}}$, the velocity difference factor is near 1 at high altitudes, decreases to 1/2 at 120 km, and further decreases at lower altitudes.

6. Conclusions

In this study we take a fluid approach to analyze a system containing ions, electrons, and neutral particles. This system can be used to describe the system of the magnetosphere, ionosphere, and thermosphere. It covers a very broad range of the parameter regime, ranging from weakly ionized dense collisional gas to fully ionized tenuous collisionless plasma. We analyze the electron and ion momentum equations in the plasma frame for a uniform medium in steady state. In particular, we include the electron-ion collisions self-consistently and treat the ion and electron equations in a completely coupled manner. We derive the complete Ohm's law and the plasma momentum equation for the system. The form of Ohm's law is in the plasma frame, similar to the generalized Ohm's law derived from a two-fluid system, while including three inter-species collisions. Ohm's law is different from the conventional three-fluid collisional Ohm's law, which is expressed in the neutral wind frame, in both its form and the expressions of the conductivities. The new form can retrieve the ideal-MHD frozen-in condition in the collisionless limit and the generalized Ohm's law when the neutral-plasma collisions are neglected. It describes continuously the transition from collisionless, to electron-ion collision dominant, to electron-ion and neutral-plasma collisions coexisting, and to neutral-plasma collision dominant processes and hence covers the full range of altitude from the magnetosphere, to ionosphere and thermosphere, as well as other space/astronomical plasma environments. The conventional three-fluid collisional Ohm's law can be derived by combining the new form of Ohm's law with the momentum equation. The conventional conductivities may be retrieved with a few approximations, which may hold well in Earth's environment, but some may be poor in other plasma environments.

Application of the new form of Ohm's law to Earth's environment shows that in the magnetosphere-ionosphere coupling, the ionosphere needs to be treated as a separate entity from the magnetosphere instead of the inner boundary of the magnetosphere. The ionospheric velocity is different from the magnetospheric velocity. The relationship between the magnetospheric electric field and ionospheric velocity can be derived by combining Ohm's laws in the plasma frame and in the neutral wind frame, indicating that the two regions are coupled primarily by the electric field. The ionosphere is coupled with the thermosphere by the neutral-plasma collisions, as described by the plasma momentum equation.

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References

Gombosi, T. I., *Gaskinetic Theory*, Cambridge Univ. Press, New York, 1994.

Gombosi, T. I., *Physics of the Space Environment*, Cambridge Univ. Press, New York, 1998.

Kelley, M. C., *The Earth's Ionsphere: Plasma Physics and Electrodynamics*, Academic, San Diego, Calif., 1989.

Luhmann, J. G., Ionospheres, in *Introduction to Space Physics*, edited by M. G. Kivelson and C. T. Russell, p. 183, Cambridge Univ. Press, New York, 1995.

Richmond, A. D., Ionospheric electrodynamics, in *Handbook of Atmospheric Electrodynamics*, edited by H. Vollan, p. 249, CRC Press, Boca Raton, Fla., 1995.

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