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PARALLEL ELECTRIC FIELDS

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Abstract—A steady state one-dimensional model is used to estimate the electron current along a field line from the auroral zone to the plasmasheet when a potential difference exists between its ends. The plasmas at either end, the ionosphere and the plasmasheet, are assumed thermal. When typical experimental values are substituted into the analytic expression obtained it is found that potential differences of the order of 10 kV are required to drive the field-aligned currents sometimes observed.

One of the most controversial questions in magnetospheric physics is the importance and magnitude of parallel electric fields. The magnetospheric plasma is for most purposes collision free, the mean free path being much larger than typical length scales. The electric field then causes a large rate of change of current and changes in electric field are liable to set up plasma oscillations. Thus the plasma frequency indicates the timescale for variations. Superficially at least parallel electric fields would be expected to be associated with parallel currents and most simple-minded approaches suggest that they should be small or at least readily neutralized, though Alfvén and co-workers state that real plasmas often contain filamentary structures in which this is not so (Alfvén *et al.*, 1964).

There is ample evidence for thin concentrated layers of field aligned current (Cloutier, 1971) and some theoretical understanding of their cause (Speiser, 1965; Cowley, 1973; Willis, 1970). Field aligned currents are observed within the plasmasheet but most commonly at its outer edge (Aubry *et al.*, 1972) or at the poleward edge of the aurora where the layers typically have a thickness of about 10 km (Park and Cloutier, 1971; Choy *et al.*, 1971). Evidence for parallel electric fields in the form of auroral precipitation peaked at low pitch angles has also been found near the poleward edge of the aurora (Whalen and McDiarmid, 1972). The spectra obtained were consistent with a potential drop of about 7 kV concentrated in the last R_E or so above their rocket.

These parallel currents could be associated with waves but these would be propagating at the Alfvén speed whereas the electrons accelerate to kV energies and move faster than the Alfvén speed and so cannot stay in phase with such waves. Hence it may be fruitful to consider a quasi-static solution for the electron current. However even in the steady state the equation of momentum transfer still contains an inertial term, which could in principle balance the electric force. This would probably be at its largest when the plasma is convected across a current layer, so that following the convection the current density would rise and fall rapidly. A strictly one dimensional static formulation is possible only if we have no plasma motion across the field tubes. This will be true in the absence of convection or in cases when the current layer is moving with the convection speed. When we have convection the time taken to cross the current layer will be the same all along the field line. If the convection speed is 0·5 km/sec in the ionosphere where the sheet is 10 km thick this will be about 20 sec and in this time kV particles will cover several tens of Earth radii. So the following one-dimensional formulation may still be reasonable as long as the electric field remains roughly constant in this kind of time scale. This will be so if the field results from the cross-tail electric field which surges with a period of about 20 min.

STATIC FORMULATION

Let us consider a field line from the high latitude auroral zone to the neutral sheet, along which the magnetic field strength decreases monotonically. Observations (Hones *et al.*, 1971) lead us to believe that it is reasonable to expect a Maxwellian steady state solution far out on this field line, though possibly with a high energy tail and we will consider the thermoelectric effect with essentially thermal plasmas at different temperatures at either end of the field line, i.e. the ionosphere and the plasmasheet. We can estimate the mean free path as $10^4\, T_e^2/N_e$ cm (Alfvén *et al.*, 1964). The temperature is high in the plasmasheet, about a million deg, and the density is low, about 0·1–1 cm^{-3}, so the mean free path here is about 10^{16} cm ($10^7\, R_E$). However the mean free path falls to the order of $1 R_E$ in the ionosphere, where the temperature is about 6000°K and the density is of the order of hundreds per cm^3. Thus we can describe electron motion along this field line by means of the collisionless Boltzmann equation.

Following Grebowsky (1968) we shall consider an electric field with the potential higher at the ionospheric base level than at the plasmasheet base level and monotonic between them. This gives us five distinct categories of electron trajectories (See Appendix). At any point on the field line each of these occupies a distinct region of velocity space and using Liouville's theorem the distribution function in each of these is Maxwellian characterized by the temperature at the originating base level. This does not give the distribution function for that part of the electron population which may be trapped between the electric potential barrier and the magnetic mirror but we are mainly interested in the parallel current to which these do not contribute provided that their distribution is symmetric in $v_{\parallel}$.

The equations of the loss cones at an arbitrary point are obtained from the equations of conservation of energy and of the first adiabatic invariant in terms of the local potential and those at the two bases.

To obtain the parallel current we need to integrate $-ev_{\parallel}f(\mathbf{v})$ over all of velocity space. Assuming a distribution symmetric in $v_{\parallel}$ for the trapped electrons the only contribution to the current comes from the 'straight through' trajectories, those from the plasmasheet in velocity space region S_1 and those from the ionosphere in region E_1 which is the negative $v_{\parallel}$ counterpart of S_1. Hence we require

$$j_{\parallel} = -e\int_S v_{\parallel}(f_S - f_E)\, \mathrm{d}^3v$$

where

$$f_Y = N_Y\left(\frac{M}{2\pi kT_Y}\right)^{3/2}\exp\left\{-\frac{M}{2kT_Y}(v_{\perp}{}^2 + v_{\parallel}{}^2) + \frac{e}{kT_Y}(\phi - \phi_Y)\right\}.$$

The subscripts Y on the density N, the temperature T and the potential ϕ, refer to the values at the ionospheric, E or plasmasheet, S, bases.

On performing this integral the current density is found to be proportional to the magnetic field strength and independent of the behaviour of the electric potential as long as this does not have a significant minimum.

From the Appendix the parallel current is

$$j_{\parallel} = eB\left[N_E\sqrt{\left(\frac{kT_E}{2\pi m}\right)}g(\Delta, T_E) - N_S\sqrt{\left(\frac{kT_S}{2\pi m}\right)}\exp(\Delta/kT_S)g(\Delta, T_S)\right]$$

where

$$g(\Delta, T) = \frac{1}{B_S}\left[\exp\left(-\frac{\Delta}{kT}\right) - \frac{B_E - B_S}{B_E}\exp\left(-\frac{\Delta}{kT}\frac{B_E}{B_E - B_S}\right)\right]$$

and $\Delta = e(\phi_E - \phi_s) \geq 0$
B_Y = magnetic field strength at base Y
m = mass of the electron.

We shall take the typical polar wind flux (protons) $(B/B_E)N\sqrt{(kT/M)}$ as a convenient unit to measure the current in where $N = N_E$ by quasi-neutrality at ionosphere and $T = T_E$ if protons are in thermal equilibrium with the ionospheric electrons. Putting $t = T_S/T_E$, $n = N_S/N_E$ and $x = \Delta/kT_E$, a dimensionless potential energy drop we get

$$\frac{j_{\|}}{\frac{eB}{B_E}N_E\sqrt{\left(\frac{kT_E}{M}\right)}} = \sqrt{\left(\frac{M}{2\pi m}\right)}\left\{\left[\left(\frac{B_S - B_E}{B_S}\right)\exp\left(-xB_S/(B_E - B_S)\right) + \frac{B_E}{B_S}\right]e^{-x}\right.$$
$$\left. - n\sqrt{t}\left[\frac{B_S - B_E}{B_S}\exp\left(-xB_S/t(B_E - B_S)\right) + \frac{B_E}{B_S}\right]\right\}. \quad (1)$$

In the upper ionosphere $B_E \sim 3 \rightarrow 6 \times 10^4\gamma$ whereas near the neutral sheet $B_S \sim 10\gamma$ and t is always greater than unity. So as long as $x(B_S/B_E) \ll 1$ and $x/t(B_S/B_E) \ll 1$ we can expand and truncate the exponentials to obtain

$$\frac{j_{\|}}{\frac{eB}{B_E}N_E\sqrt{\frac{kT_E}{M}}} \approx \sqrt{\left(\frac{M}{2\pi m}\right)}\left\{\left[\frac{B_S - B_E}{B_S}\left(1 - \frac{xB_S}{B_E - B_S} + \ldots\right) + \frac{B_E}{B_S}\right]e^{-x}\right.$$
$$\left. - n\sqrt{t}\left[\frac{B_S - B_E}{B_S}\left(1 - \frac{x}{t}\frac{B_S}{B_E - B_S} + \ldots\right) + \frac{B_E}{B}\right]\right\}$$
$$\approx \sqrt{\left(\frac{M}{2\pi m}\right)}\left\{(1 + x)e^{-x} - n\sqrt{t}\left(1 + \frac{x}{t}\right)\right\}. \quad (2)$$

For no potential difference $j_{\|} \propto (1 - n\sqrt{t})$ and for infinite potential difference $j_1 \propto -n\sqrt{t} \times (B_E/B_S)$ from result (1). Therefore if $n\sqrt{t} < 1$ the current will go to zero for a non-zero potential drop. From the approximation (2) it will be seen that the current will be negative, i.e. away from the Earth, when the potential is such that

$$(1 + x)\exp(-x) < n\sqrt{t}.$$

So the highest potential difference giving no current occurs for the lowest value of $n\sqrt{t}$.

Using the observational results

$$N_S = 0{\cdot}1 \rightarrow 1\ \text{cm}^{-3} \qquad T_S = 10^{6}\,^{\circ}\text{K}$$
$$N_E = 100 \rightarrow 400\ \text{cm}^{-3} \qquad T_E = 10^3 \rightarrow 10^4\,^{\circ}\text{K}$$
$$n = 3 \times 10^{-4} \rightarrow 10^{-2} \qquad t = 10^2 \rightarrow 10^3$$

the extreme values of $n\sqrt{t}$ are 0·3 and 0·003. The corresponding dimensionless potential drop for no current is then in the range $2 \rightarrow 8$, well within the range of validity of the approximation used. We can regard (2) as giving the density ratio in terms of the null current potential difference, x_0 at a given temperature ratio t

$$n(X_0) = \frac{e^{-X_0}(1 + X_0)}{\sqrt{t}(1 + X_0/t)}.$$

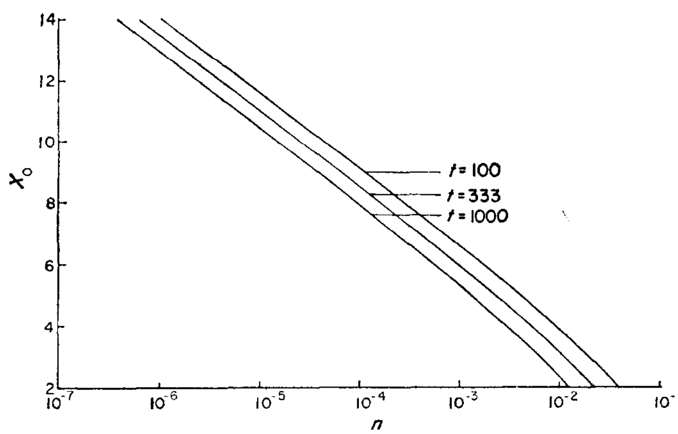

FIG. 1. GRAPH FOR CURRENT ZERO.
Density ratio against dimensionless potential drop for zero electron current.

From Fig. 1 it is easily seen that even the most extreme probable density ratio, $n = 10^{-5}$ ($N_S = 10^{-2}\,\text{cm}^{-3}$, $N_E = 10^3\,\text{cm}^{-3}$) gives null current potential difference between 10·5 and 11·5 in dimensionless units for temperature ratios between 10^2 and 10^3. Beyond this zero increasing potential exponentially kills off the ionospheric contribution (when $(1 + x)\,e^{-x} \ll n\sqrt{t}$) and the current profile goes linearly like $-n\sqrt{t}(1 + x/t)$ until the second approximation, $x \ll B_E t/B_S$ is no longer valid, whereupon the current rapidly limits (Figs. 2 and 3). It is conceivable however that a minimum of electrostatic potential could be set up somewhere along the field line, effectively concentrating the field towards the ionospheric end. When we do have a significant minimum between the bases six distinct electron trajectories are possible, reflection of plasmasheet electrons now being possible at the potential barrier as well as at the ionospheric magnetic mirror. However at the minimum itself we have only three categories and we get the following simple expressions for the density and the electron flux towards the Earth at the minimum, taking the zero of electrostatic potential there for convenience.

$$N_{\min} = N_S \exp(-e\phi_S/kT_S)[1 - \tfrac{1}{2}F^*(T_S)] + N_E \exp(-e\phi_E/kT_E)\tfrac{1}{2}F^*(T_E)$$

$$J_{\min} = N_S \exp(-e\phi_S/kT_S)\sqrt{\left(\frac{kT_S}{2\pi m}\right)}F(T_S) - N_E \exp(-e\phi_E/kT_E)\sqrt{\left(\frac{kT_E}{2\pi m}\right)}F(T_E)$$

where

$$F^*(T) = 1 - \sqrt{\left(\frac{B_E - B_{\min}}{B_E}\right)}\exp\left(-\frac{e\phi_E}{kT}\frac{B_{\min}}{B_E - B_{\min}}\right)$$

and

$$F(T) = 1 - \frac{B_E - B_{\min}}{B_E}\exp\left(-\frac{e\phi_E}{kT}\frac{B_{\min}}{B_E - B_{\min}}\right).$$

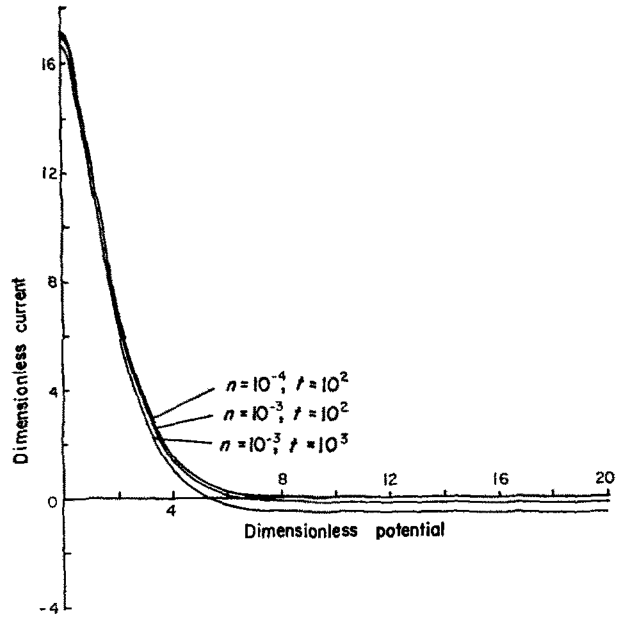

Fig. 2. Current–potential relation near zero current.
Both in dimensionless units (see text). Density ratios lowei than 10^{-4} omitted for clarity. Current positive towards the Earth.

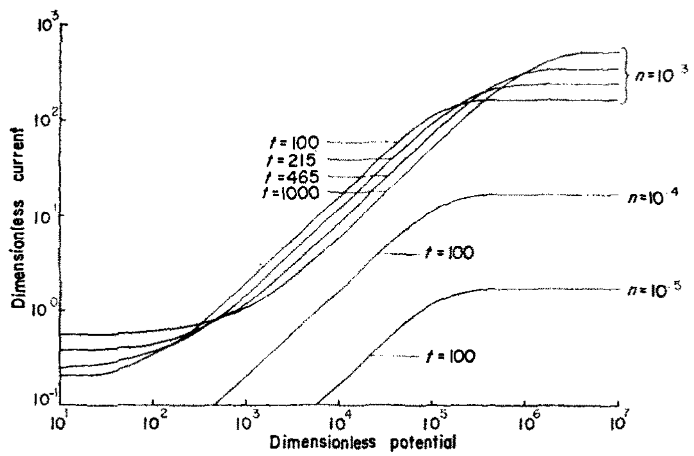

Fig. 3. Current–potential relation up to limiting.
Showing variation proportional to density ratio for a range of temperature ratios. Current away from the Earth.

Define a dimensionless current C by referring $J_{\min}$ to the unit of polar wind flux i.e.

$$C = -\frac{eJ_{\min}}{e\dfrac{B_{\min}}{B_E}N_E\sqrt{\left(\dfrac{kT_E}{M}\right)}}$$

$$C = \sqrt{\left(\frac{M}{2\pi m}\right)}\frac{B_E}{B_{\min}}\left\{\exp\left(-e\phi_E/kT_E\right)F(T_E) - \frac{N_S}{N_E}\exp\left(-e\phi_S/kT_S\right)\sqrt{\left(\frac{T_S}{T_E}\right)}F(T_S)\right\}$$

which is the same result as in (1) but with the following replacements

$$\frac{N_S \exp(-e\phi_S/kT_S)}{N_E} \quad \text{for } n$$

$$B_{\min} \quad \text{for } B_S$$

$$\frac{e\phi_E}{kT_E} \quad \text{for } x.$$

So now the limiting negative current, flux towards the Earth, given by

$$-\sqrt{\left(\frac{M}{2\pi m}\right)}\frac{N_S \exp(-e\phi_S/kT_S)}{N_E}\sqrt{\left(\frac{T_S}{T_E}\right)}\frac{B_E}{B_{\min}}$$

is a function of the position of the minimum and so if this is close to the Earth a substantial reduction in the maximum possible outward current results (Fig. 4). The variation with n and t is just the same as before (Figs. 2 and 3) except that n can now be reduced by a deepening of the minimum corresponding to an increased (positive) ϕ_S. Near zero current the current–potential profile becomes independent of the position of the minimum and so is almost exactly the same as for no minimum.

CONCLUSIONS

We have obtained an estimate of the current carried by the electrons along field lines, assuming that their higher mobility makes their's the dominant contribution. The existence of an electric field directed out of the ionosphere will have no effect on the normal polar wind proton flux which can be regarded as carrying a current of 0·4 $eN_E\sqrt{(kT_E/M)}$. We can use the results of this calculation to infer the potential drop causing a given current density, though without simultaneous measurements of the base level plasma conditions this will not be much better than an order of magnitude estimate.

If we take the ionosphere base temperature and density to be 6000°K and 10^3 cm^{-3} respectively our dimensionless current unit has magnitude $1{\cdot}12 \times 10^{-6}$ A/m^2 corresponding to a flux of 7×10^8 particles/cm^2 sec. The observations of transverse magnetic perturbation give the current intensity in the auroral zone ionosphere in the range 0·02 → 0·7 A/m (Zmuda *et al.*, 1970) and extrapolations of data from OGO-5 in the tail give values between 0·2 and 0·5 A/m (Aubry *et al.*, 1972). To get the current density we must know the thickness of the current layer, a value of about 10 km probably being reasonable although arguably they are sometimes much thicker than this (Armstrong and Zmuda, 1970). Thus the observed current densities are in the range $2 \times 10^{-6} \rightarrow 7 \times 10^{-5}$ A/m^2 or 1–65 in our dimensionless units. This is consistent with Injun-5 observations of peak isotropic downward fluxes of about 10^{10}/cm^2 sec sr (Frank and Ackerson, 1971).

Taking the density ratio, $n = 10^{-3}$, the temperature ratio, $t = 10^2$ and the magnetic field strength ratio 10^3 we see from Fig. 3 that a potential energy difference of $1{\cdot}2 \times 10^4 KT_E$ is required to drive a dimensionless current of 20, so we need a voltage drop of 6 kV in order to have a current density at the ionosphere directed away from the Earth of 2×10^{-5} A/m^2. A similar magnitude of current density directed towards the Earth results if there is zero voltage drop (see Fig. 2).

From Fig. 4 we see that by bringing the minimum of potential towards the Earth it is possible to reduce the current obtained at a given potential provided that this is significantly

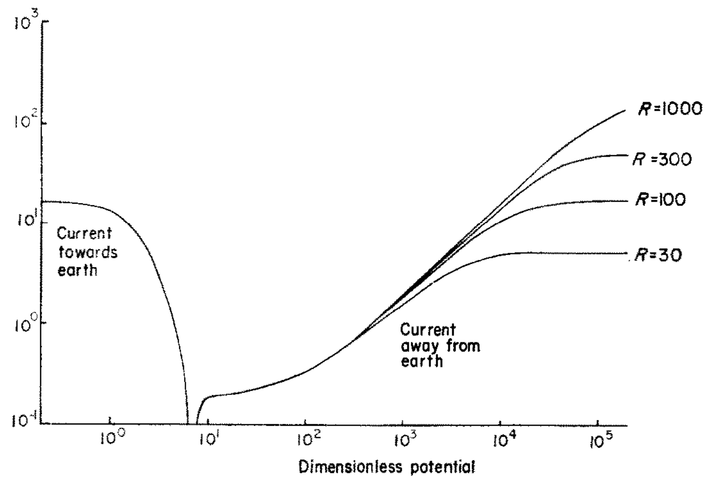

FIG. 4. CURRENT–POTENTIAL RELATION FOR DECREASING VALUES OF $R = B_E/B_{min}$, DENSITY RATIO IS 10^{-3} AND TEMPERATURE RATIO 10^2.

greater than that for no current at the prevailing values of the temperature and density ratios. The potential is controlled by the proton density and density observation would greatly help to decide on the relevance of the models considered here.

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APPENDIX

Loss-cones

The equations of conservation of energy and first adiabatic invariant give

$$v_\perp'^2 + v_\parallel'^2 - \frac{2e}{m}\phi' = v_\perp^2 + v_\parallel^2 - \frac{2e}{m}\phi$$

and $v_\perp'^2/B' = v_\perp^2/B$ where primes denote values at a base, unprimed quantities at an arbitrary point on the field line. For a trajectory to pass through the appropriate base we need $v_\parallel'^2 > 0$ which gives

$$\frac{2e}{m}(\phi' - \phi) + v_\perp^2\left(1 - \frac{B'}{B}\right) + v_\parallel^2 > 0.$$

So trajectories passing though the ionosphere base must occupy a region A of velocity space satisfying this equation with ϕ_E for ϕ' and B_E for B' and those through the plasmasheet base a region B where ϕ_S, B_S are read for ϕ', B' respectively.

Now if $\phi_E > \phi > \phi_S$ and $B_E > B > B_S$ the region A lies between the hyperbola $C_E^2 v_\perp^2 - v_\parallel^2 = D_E^2$ and the $v_\perp = 0$ axis and the region B lies outside the ellipse $C_S^2 v_\perp^2 + v_\parallel^2 = D_S^2$ where

$$C_S^2 = 1 - B_S/B \qquad D_S^2 = \frac{2e}{m}(\phi - \phi_S)$$

$$C_E^2 = B_E/B - 1 \qquad D_E^2 = \frac{2e}{m}(\phi_E - \phi).$$

In the case of a monotonic potential

The straight through trajectories	$E_1, S_1 \subseteq A \cap B$
the trapped trajectories	$T \subseteq \bar{A} \cap \bar{B}$
from ionosphere not reaching sheet	$E_2 \subseteq A \cap \bar{B}$
from sheet not reaching ionosphere	$S_2 \subseteq \bar{A} \cap B$.

Where the not-bar indicates the complement and these five disjoint regions account for all of velocity space. If, however, there is a minimum, i.e. a point where $\phi < \phi_S$, region B becomes all of velocity space at that point and $\bar{B}$ is null. Thus there are the straight through trajectories, E_1 and S_1, in region A and trajectories from the plasmasheet not reaching the ionosphere, S_2, in its complement obviously occupying all of velocity space.

The current integral

The parallel current is given by

$$j_\parallel = -e\int_\Sigma v_\parallel(f_S - f_E)\,\mathrm{d}^3v$$

where Σ in the region $A \cap B$ for $v_\parallel \geq 0$
and

$$f_Y = N_Y\left(\frac{m}{2\pi kT_Y}\right)^{3/2}\exp\left\{-\frac{m}{2kT_Y}(v_\perp^2 + v_\parallel^2) + \frac{e}{kT_Y}(\phi - \phi_Y)\right\}.$$

Thus

$$j_{\parallel} = -e\{\exp [e(\phi - \phi_S)/kT_S]N_S I(T_S) - \exp [e(\phi - \phi_E)/kT_E]N_E I(T_E)\} \tag{3}$$

where

$$I(T) = \left(\frac{m}{2\pi kT}\right)^{3/2} 2\pi \iint_{\Sigma} v_{\parallel} v_{\perp} \exp(-\tfrac{1}{2}mv^2/kT)\, \mathrm{d}v_{\parallel}\, \mathrm{d}v_{\perp}.$$

In general the loss cones intersect at $v_{\parallel} = V$ where $V^2 = (C_E^2 D_S^2 - C_S^2 D_E^2)/(C_E^2 + C_S^2)$ so for $V < v_{\parallel} < D_S$ the limits of $v_{\perp}$ will be $\sqrt{([D_S^2 - v_{\parallel}^2]/C_S^2)}$ to $\sqrt{([v_{\parallel}^2 + D_E^2]/C_E^2)}$ and for $v_{\parallel} > D_S$ from 0 to $\sqrt{([v_{\parallel}^2 + D_E^2]/C_E^2)}$. To simplify the integration put $A^2 = m/2KT$, $x = Av_{\perp}$, $y = Av_{\parallel}$.

Thus

$$\begin{aligned} I(T) &= \frac{2}{A\sqrt{\pi}}\left\{\int_{AD_S}^{\infty} y \exp(-y^2)\,\mathrm{d}y \int_0^{(y^2+A^2D_E^2)^{1/2}/C_E} \mathrm{d}[-\exp(-x^2)/2]\right. \\ &\quad \left. + \int_{AV}^{AD_S} y \exp(-y^2)\,\mathrm{d}y \int_{(A^2D_S^2-y^2)^{1/2}/C_S}^{(y^2+A^2D_E^2)^{1/2}/C_E} \mathrm{d}(-\exp(-x^2)/2)\right\} \\ &= \frac{1}{2A\sqrt{\pi}}\left\{\exp(-A^2D_S^2)\left[1 - \frac{1}{1-(1/C_S^2)}\right]\right. \\ &\quad - \frac{1}{1+(1/C_E^2)} \exp\left(-\frac{A^2[D_E^2 + V^2(C_E^2+1)]}{C_E^2}\right) \\ &\quad \left. + \frac{1}{1-(1/C_S^2)} \exp\left(-\frac{A^2[D_S^2 + (C_S^2-1)V^2]}{C_S^2}\right)\right\} \\ &= \sqrt{\left(\frac{kT}{2\pi m}\right)} \exp\left[\frac{e(\phi_E - \phi)}{kT}\right] B\left\{\frac{1}{B_S} \exp\left(\frac{e(\phi_S - \phi_E)}{kT}\right)\right. \\ &\quad \left. + \left(\frac{1}{B_E} - \frac{1}{B_S}\right) \exp\left(\frac{eB_E(\phi_S - \phi_E)}{kT(B_E - B_S)}\right)\right\} \end{aligned}$$

Defining $\Delta = e(\phi_E - \phi_S)$ we can write

$$I(T) = \sqrt{\left(\frac{kT}{2\pi m}\right)} \exp[e(\phi_E - \phi)/kT] B g(\Delta, T)$$

where

$$g(\Delta, T) = \frac{1}{B_S} \exp(-\Delta/kT)\left\{1 - \frac{B_E - B_S}{B_E} \exp\left[-\Delta/kT\left(\frac{B_E}{B_S} - 1\right)\right]\right\}$$

and thus the parallel current is

$$j_{\parallel} = -eB\left\{\exp(\Delta/kT_S) N_S \sqrt{\left(\frac{kT_S}{2\pi m}\right)} g(\Delta, T_S) - N_E \sqrt{\left(\frac{kT_E}{2\pi m}\right)} g(\Delta, T_E)\right\}.$$

If there is a minimum of potential on the field line it is logical to take this as the zero of potential. The current is given by the same formula (3) but the integral now involved in

$$I(T) = \left(\frac{m}{2\pi kT}\right)^{3/2} 2\pi \int_0^{\infty} \mathrm{d}v_{\parallel} \int_0^{\sqrt{(v_{\parallel}^2 + D_E^2)}/C_E} \mathrm{d}v_{\perp} v_{\parallel} v_{\perp} e^{-1/2mv^2/kT}$$

where D_E^2 is now $(2e/m)\phi_E$ and $C_E^2 = (B_E/B_{\min}) - 1$

$$I(T) = \frac{2}{\sqrt{(\pi A^2)}} \int_0^\infty \mathrm{d}y y \exp(-y^2) \int_0^{\sqrt{(y^2+A^2D_E^2)}/C_E} x \exp(-x^2)\,\mathrm{d}x$$

substituting $m/2kT = A^2$ and $A^2V_{\parallel}{}^2 = y^2$, $A^2V_{\perp}{}^2 = x^2$

$$I(T) = \frac{2}{A\sqrt{\pi}} \int_0^\infty \mathrm{d}y y \exp(-y^2) \frac{1}{2}\left[1 - \exp\left(\frac{-(y^2 + A^2D_E^2)}{C_E^2}\right)\right]$$

$$= \sqrt{\left(\frac{kT}{2\pi m}\right)} F(T)$$

where

$$F(T) = 1 - \frac{B_E - B_{\min}}{B_E} \exp\left[-e\phi_E/kT\left(\frac{B_E}{B_{\min}} - 1\right)\right].$$

We can calculate the density at the minimum.

$$N_{\min} = \int_{E1} f_E\,\mathrm{d}^3v + \int_{E1} f_S\,\mathrm{d}^3v$$

$$= \int_{E1} [f_E - f_S]\,\mathrm{d}^3v + \int_{\substack{\text{all velocity}\\ \text{space}}} f_S\,\mathrm{d}^3v$$

$$= N_E \exp(-e\phi_E/kT_E) I^*(T_E) - N_S \exp(-e\phi_S/kT_S) I^*(T_S)$$
$$+ N_S \exp(-e\phi_S/kT_S)$$

where

$$I^*(T) = \left(\frac{m}{2\pi kT}\right)^{3/2} 2\pi \int_0^\infty \mathrm{d}v_\parallel \int_0^{\sqrt{(v_\parallel{}^2+D_E^2)}/C_E} \mathrm{d}v_\perp v_\perp \exp(-\tfrac{1}{2}m\mathrm{v}^2/kT)$$

making the same substitution we see that

$$I^*(T) = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{d}y \exp(-y^2)\left[1 - \exp\left(-\frac{y^2 + A^2D_E^2}{C_E^2}\right)\right]$$

$$= \frac{1}{2}\left\{1 - \sqrt{\left(\frac{B_E - B_{\min}}{B_E}\right)} \exp\left[-e\phi_E/kT\left(\frac{B_E}{B_{\min}} - 1\right)\right]\right\}$$

$$= \tfrac{1}{2}F^*(T) \quad \text{defining } F^*(T).$$